11 Rational Equations and Functions

"Descartes, in your homework problem, you are being chased by a cat-eating hyena."

"Both of you must stay on the graph of $y = 1/x$. The safe zone is the x-axis."

"Can you ever reach the safe zone? Explain your reasoning."

"Descartes, I am keeping track of how many doggy treats my owner gives me each day."

"I am finding that my happiness is directly proportional to the day of the week."

Connections to Previous Learning

- Apply properties of operations to generate equivalent expressions.
- Solve simple one-variable equations to solve real-life and mathematical problems.
- Use variables to represent two quantities that change in relationship to one another.

- Use and simplify algebraic expressions to solve problems.
- Analyze and solve linear equations to solve real-life and mathematical problems.

- Create equations that describe numbers or relationships.
- Add, subtract, multiply, and divide rational expressions.
- Create and solve rational equations in one variable, and use them to solve problems.

Pacing Guide for Chapter 11

Chapter Opener	1 Day
Section 1	2 Days
Section 2	3 Days
Section 3	1 Day
Study Help / Quiz	1 Day
Section 4	1 Day
Section 5	1 Day
Section 6	1 Day
Section 7	1 Day
Chapter Review / Chapter Tests	2 Days
Total Chapter 11	14 Days
Year-to-Date	151 Days

Chapter Summary

Section	Common Core State Standard	
11.1	Learning	A.REI.10,
11.2	Learning	A.REI.10 ★, F.BF.4a ★
11.3	Learning	A.SSE.2
11.4	Learning	A.SSE.2
11.5	Learning	A.SSE.2
11.6	Learning	A.SSE.2 ★
11.7	Applying	A.CED.1 ★

★ Teaching is complete. Standard can be assessed.

Technology for the Teacher

BigIdeasMath.com
Chapter at a Glance
Complete Materials List
Parent Letters: English and Spanish

Common Core State Standards

5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.
6.NS.1 Interpret and compute quotients of fractions, ...
7.RP.3 Use proportional relationships to solve multistep ratio . . . problems.

Additional Topics for Review

- Domain and range of a function
- Relations and functions
- Polynomials
- Long division

Try It Yourself

1. 1
2. $\frac{1}{2}$
3. $\frac{5}{3}$
4. $\frac{5}{16}$
5. 3
6. 2
7. 7.8

Record and Practice Journal Fair Game Review

1. $\frac{7}{3}$
2. $\frac{1}{8}$
3. $-\frac{7}{10}$
4. $\frac{9}{4}$
5. $\frac{1}{8}$
6. $\frac{5}{12}$
7. $\frac{3}{5}$
8. 10
9. $\frac{13}{8}$ mi
10. 6 servings
11. $d = 4$
12. $m = 15$
13. $a = 27$
14. $k = 3.2$
15. $x = 11.4$
16. $c = 12.75$
17. $212.50

Math Background Notes

Vocabulary Review

- Fractions
- Proportions
- Cross Products Property

Evaluating Expressions with Fractions

- Students should be able to add, subtract, multiply, and divide fractions with like and unlike denominators.
- When adding or subtracting two fractions that share a common denominator, simply add or subtract their numerators and keep the common denominator. If two fractions do not have a common denominator, students will need to find one.
- You may want to review finding the Least Common Multiple between two numbers. Remind students that finding the Least Common Multiple among the denominators will produce the least common denominator used to add or subtract fractions.
- You may want to review how to rename fractions using the least common denominator.
- Remind students that the rules for multiplying and dividing fractions are different from the rules for adding and subtracting fractions. Multiplying and dividing fractions does not require a common denominator.
- **Teaching Tip:** Most students will remember the process to divide fractions. If your students are comfortable with the process, encourage them to describe it using math vocabulary. Instead of, "change the sign and flip the second fraction," encourage "multiply by the reciprocal of the divisor."

Solving Proportions

- Students should be able to solve proportions.
- You may wish to review the Cross Products Property. Remind students that this property is unique as it can only be used in proportions.
- **Teaching Tip:** Help students to visualize the Cross Products Property by drawing the X as you work through the problem. Example:

$$\frac{2}{5} \quad \frac{x}{15}$$

$$30 = 5x$$

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Record and Practice Journal • Fair Game Review Skills Review Handbook Lesson Tutorials	Game Closet at *BigIdeasMath.com* Start the next section

What You Learned Before

"Your namesake, René Descartes, believed in rational thinking."

Evaluating Expressions with Fractions (5.NF.1, 6.NS.1)

Example 1 Find $\frac{1}{10} + \frac{3}{5}$.

$$\begin{aligned}\frac{1}{10} + \frac{3}{5} &= \frac{1}{10} + \frac{6}{10}\\ &= \frac{1+6}{10}\\ &= \frac{7}{10}\end{aligned}$$

Example 2 Find $\frac{4}{3} \div \frac{5}{3}$.

$$\begin{aligned}\frac{4}{3} \div \frac{5}{3} &= \frac{4}{3} \cdot \frac{3}{5}\\ &= \frac{4 \cdot \cancel{3}}{\cancel{3} \cdot 5}\\ &= \frac{4}{5}\end{aligned}$$

Try It Yourself

Evaluate the expression.

1. $\frac{1}{6} + \frac{5}{6}$ **2.** $\frac{2}{3} - \frac{1}{6}$ **3.** $\frac{5}{2} \cdot \frac{2}{3}$ **4.** $\frac{5}{6} \div \frac{8}{3}$

Solving Proportions (7.RP.3)

Example 3 Solve $\frac{4}{x} = \frac{5}{12}$.

$\frac{4}{x} = \frac{5}{12}$ Write the proportion.

$4 \cdot 12 = x \cdot 5$ Use the Cross Products Property.

$48 = 5x$ Multiply.

$9.6 = x$ Divide each side by 5.

Try It Yourself

Solve the proportion.

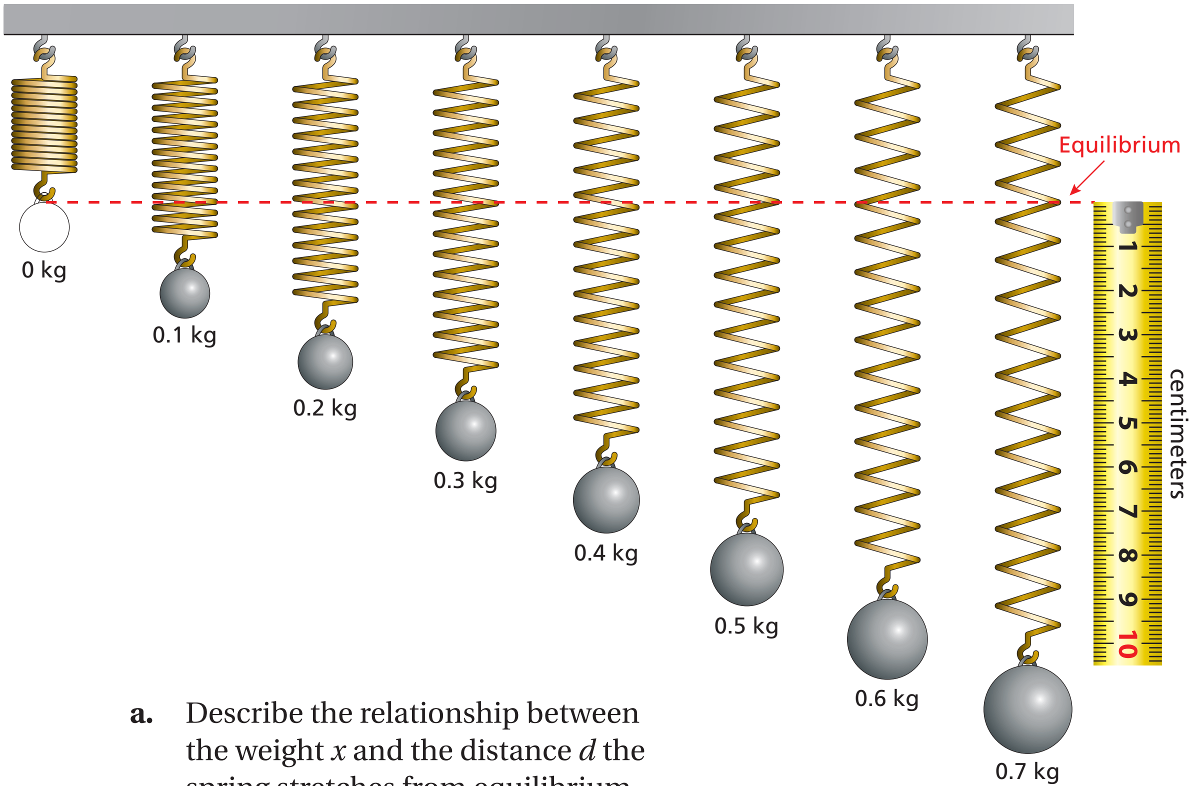
5. $\frac{4}{6} = \frac{2}{x}$ **6.** $\frac{3}{12} = \frac{w}{8}$ **7.** $\frac{15}{y} = \frac{25}{13}$

11.1 Direct and Inverse Variation

Essential Question How can you recognize when two variables vary directly? How can you recognize when they vary inversely?

1 ACTIVITY: Recognizing Direct Variation

Work with a partner. You hang different weights from the same spring.

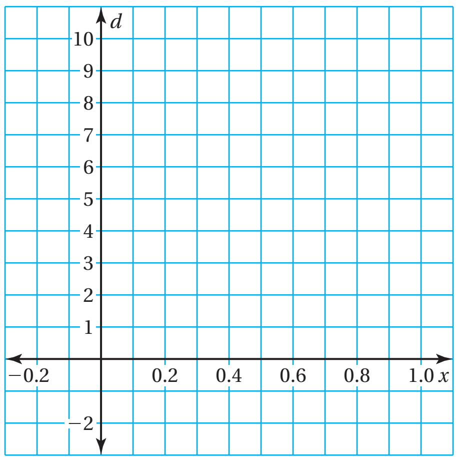


a. Describe the relationship between the weight x and the distance d the spring stretches from equilibrium. Explain why the distance is said to vary *directly* with the weight.

b. Graph the relationship between x and d. What are the characteristics of the graph?

c. Write an equation that represents d as a function of x.

d. In physics, the relationship between d and x is described by Hooke's Law. How would you describe Hooke's Law?



COMMON CORE

Direct and Inverse Variation

In this lesson, you will
- identify direct and inverse variation.
- write and graph direct and inverse variation equations.

Learning Standard
A.REI.10

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP2 Reason Abstractly and Quantitatively:** Students will recognize patterns as they work with direct variation and inverse variation. It is important that students be able to describe the quantitative relationship between x and y.

Motivate

- Collect three items that have different lengths. Each item should have a length that is approximately a whole number of inches.
- Measure each of the items in inches, and again in centimeters. Record the results in a table.

Note pad	Travel mug	Paper	?
3 in.	6 in.	11 in.	
7.62 cm	15.24 cm	27.94 cm	

? "If I measure another item and tell you its length in inches, can you tell me its length in centimeters? Explain." yes; Listen for students to describe the process using the relationship 1 inch = 2.54 centimeters.

- Pretend to measure another item and say it is 4 inches long. Add it to the table and ask students for the length in centimeters.

? "If you plot the points (inches, centimeters), what type of function is represented by the graph?" linear

Activity Notes

Activity 1

- In this activity, students need to read lengths from the tape measure displayed at the right.
- **MP6 Attend to Precision:** To improve accuracy, students could use a straightedge, keeping it parallel to the dotted line.
- You could have a general discussion with students about what equilibrium means in this context.
- Each 0.1 kilogram adds approximately 1.5 centimeters to the length.
- When students have finished, ask them to share their responses in part (a). It is likely that students will mention the constant rate of change.
- When describing the graph in part (b), students should mention the slope and that the line passes through the origin.

Common Core State Standards

A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Previous Learning

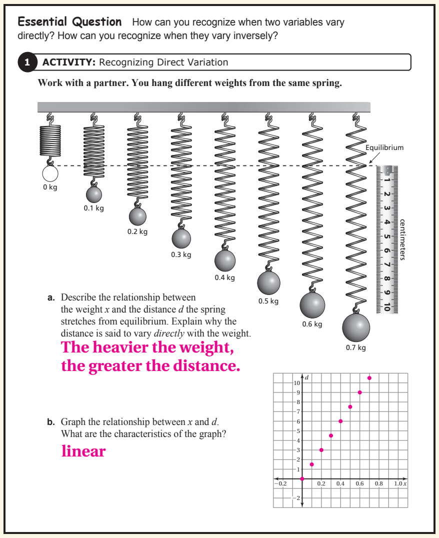
Students should know how to graph linear and nonlinear functions.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.1 Record and Practice Journal



Essential Question How can you recognize when two variables vary directly? How can you recognize when they vary inversely?

1 ACTIVITY: Recognizing Direct Variation

Work with a partner. You hang different weights from the same spring.

a. Describe the relationship between the weight x and the distance d the spring stretches from equilibrium. Explain why the distance is said to vary *directly* with the weight.
The heavier the weight, the greater the distance.

b. Graph the relationship between x and d. What are the characteristics of the graph?
linear

Differentiated Instruction

Advanced

Challenge students to look for symmetry in graphs that represent inverse variation. They should see that the graph is symmetric about the line $y = x$. This occurs when exchanging x and y in the original equation produces an equivalent equation. What other equations have graphs that are symmetric about the line $y = x$?

$y = x$ and $y = -x$

11.1 Record and Practice Journal

c. Write an equation that represents d as a function of x.

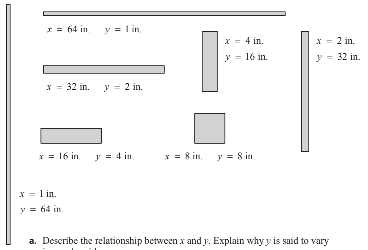
$d = 15x$

d. In physics, the relationship between d and x is described by Hooke's Law. How would you describe Hooke's Law?

The distance a spring stretches from equilibrium is a constant multiple of the amount of weight applied to it.

2 **ACTIVITY:** Recognizing Inverse Variation

Work with a partner. The area of each rectangle is 64 square inches.

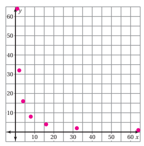


a. Describe the relationship between x and y. Explain why y is said to vary inversely with x.

The greater the length of x, the smaller the length of y.

b. Graph the relationship between x and y. What are the characteristics of the graph?

nonlinear



c. Write an equation that represents y as a function of x.

$y = \frac{64}{x}$

What Is Your Answer?

3. IN YOUR OWN WORDS How can you recognize when two variables vary directly? How can you recognize when they vary inversely?

directly: one variable is a constant multiple of the other
inversely: one variable is a constant multiple of the inverse of the other

4. Does the flapping rate of a bird's wings vary directly or inversely with the length of its wings? Explain your reasoning.

inversely

Laurie's Notes

Activity 2

- Work through the activity as shown.
- If students have difficulty writing the equation in part (c), ask them what they know about the product xy. This should lead them to write $xy = 64$, which they can solve for y.
- **Common Error:** Students may believe the graph represents exponential decay. Point out that there is no y-intercept. After writing the equation in part (c), they should also recognize that x is not an exponent.

What Is Your Answer?

- You want students to recognize that with direct variation, x and y increase (or decrease) proportionally. With inverse variation, x increases as y decreases (or vice versa), and x and y are inversely proportional. Students may say that if x doubles, y is divided by 2.

Closure

- Explain why the situation in the Motivate (measuring in inches and then centimeters) involves direct variation. The measurements change proportionally.

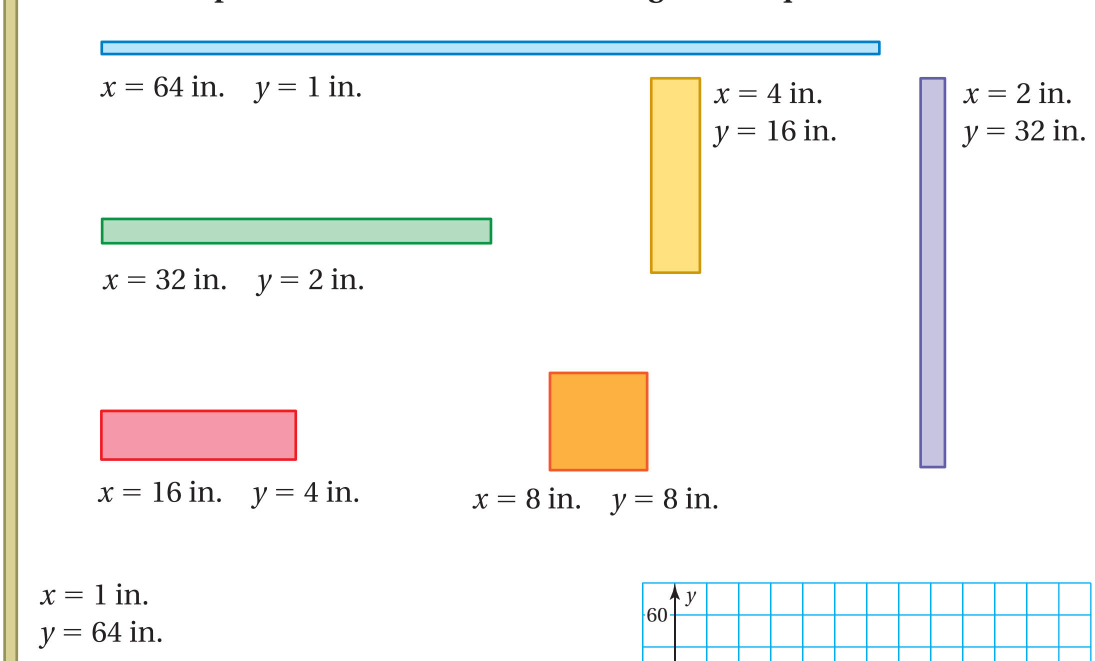
2 ACTIVITY: Recognizing Inverse Variation

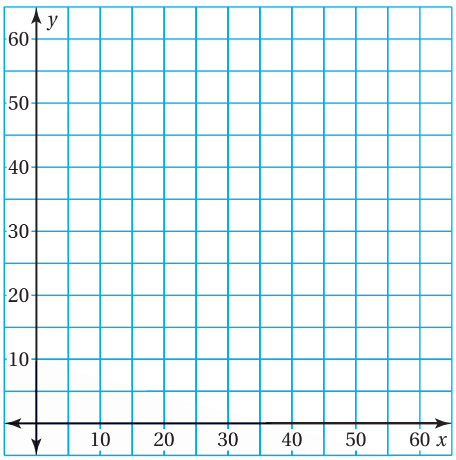
Math Practice 6

Calculate Accurately

How can you verify that your graph and equation represent the relationship between x and y?

Work with a partner. The area of each rectangle is 64 square inches.





a. Describe the relationship between x and y. Explain why y is said to vary inversely with x.

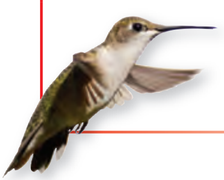
b. Graph the relationship between x and y. What are the characteristics of the graph?

c. Write an equation that represents y as a function of x.

What Is Your Answer?

3. **IN YOUR OWN WORDS** How can you recognize when two variables vary directly? How can you recognize when they vary inversely?

4. Does the flapping rate of a bird's wings vary directly or inversely with the length of its wings? Explain your reasoning.



Practice Use what you learned about direct and inverse variation to complete Exercises 3 and 4 on page 547.

11.1 Lesson

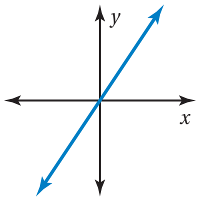
Check It Out
Lesson Tutorials
BigIdeasMath.com

Key Vocabulary
direct variation, *p. 544*
inverse variation, *p. 544*

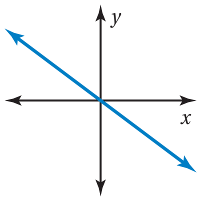
Key Ideas

Direct Variation

Two quantities x and y show **direct variation** when $y = kx$, where k is a nonzero constant.



$y = kx,\ k > 0$

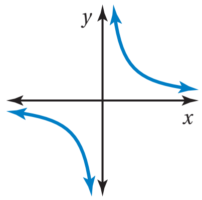


$y = kx,\ k < 0$

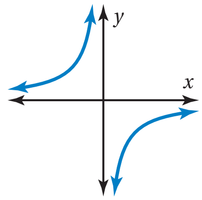
The ratio $\frac{y}{x}$ is constant.

Inverse Variation

Two quantities x and y show **inverse variation** when $y = \frac{k}{x}$, where k is a nonzero constant.



$y = \frac{k}{x},\ k > 0$



$y = \frac{k}{x},\ k < 0$

The product xy is constant.

Study Tip

The constant k is called the *constant of proportionality* or the *constant of variation*.

EXAMPLE 1 Identifying Direct and Inverse Variation

Tell whether x and y show *direct variation*, *inverse variation*, or *neither*. Explain your reasoning.

a.

x	1	2	3	4
y	5	10	15	20

The products xy are not constant. So, the table does not show inverse variation.

Check each ratio $\frac{y}{x}$: $\quad \frac{5}{1} = 5, \quad \frac{10}{2} = 5, \quad \frac{15}{3} = 5, \quad \frac{20}{4} = 5$

∴ The ratios are constant. So, x and y show direct variation.

b. $4xy = -4$

$y = -\frac{1}{x}$ Divide each side by $4x$.

∴ The equation is of the form $y = \frac{k}{x}$. So, x and y show inverse variation.

On Your Own

Now You're Ready
Exercises 3–6

Tell whether x and y show *direct variation*, *inverse variation*, or *neither*. Explain your reasoning.

1.

x	1	2	3	4
y	24	12	8	6

2. $y = 3x + 1$

Multi-Language Glossary at BigIdeasMath.com

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students recognized direct and inverse variation. (MP2, MP6)
- **Today:** Students will write and graph direct and inverse variation equations.

Motivate

- **Story Time:** Tell students they have \$48 to spend at a store. They must spend all of their money and they can only purchase one type of item.
- Indicate that the purchase prices of all items are factors of 48.
- Give students 48 seconds to decide how to spend the \$48. For example, they could purchase forty-eight \$1 items or eight \$6 items.
- Have a discussion about the "purchases." Students should recognize that the more expensive the item, the fewer you can purchase. The cheaper the item, the more you can purchase.

Lesson Notes

Key Ideas

- Write the Key Ideas, defining direct variation and inverse variation.
- Discuss the constant k and the affect of $k > 0$ and $k < 0$.
- ? "Direct variation is a special case of what type of function?" linear
- Solve each equation for k to convince students that the ratio y to x is constant in a direct variation equation and the product of x and y is constant in an inverse variation equation.
- Discuss the vocabulary in the Study Tip. Use both "constant of variation" and "constant of proportionality" when working through examples.

Example 1

- Write the table of values in part (a).
- ? "How can you tell from the table whether x and y show direct variation, inverse variation, or neither?" Check for constant ratios (y to x) or constant products (xy).
- Remind students that they must check all ordered pairs, not just a few.
- Write the equation in part (b).
- ? **MP7 Look for and Make Use of Structure:** "How can you tell from the equation whether x and y show direct variation, inverse variation, or neither?" Solve for y and interpret the result.

On Your Own

- **Think-Pair-Share:** Students should read each question independently and then work in pairs to answer the questions. When they have answered the questions, the pair should compare their answers with another group and discuss any discrepancies.

Goal Today's lesson is writing and graphing **direct** and **inverse variation** equations.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Tell whether x and y show *direct variation, inverse variation,* or *neither*. Explain your reasoning.

a.

x	1	2	3	4
y	3	$\frac{3}{2}$	1	$\frac{3}{4}$

inverse variation; The products xy are constant.

b. $2y = 5x$

direct variation; The equation can be written as $y = kx$.

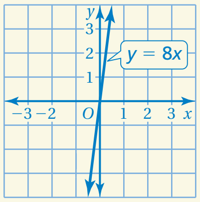
On Your Own

1. inverse variation; The products xy are constant.
2. neither; The equation cannot be written as $y = kx$ or $y = \frac{k}{x}$.

Extra Example 2

The variable y varies directly with x. When $x = 4$, $y = 32$. Write and graph a direct variation equation that relates x and y.

$y = 8x$

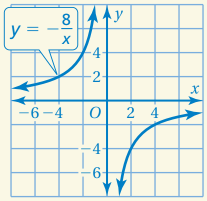


Extra Example 3

The variable y varies inversely with x. When $x = 2$, $y = -4$.

a. Write an inverse equation that relates x and y. $y = -\dfrac{8}{x}$

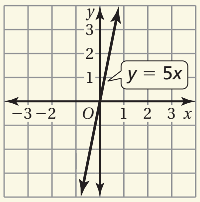
b. Graph the inverse variation equation. Describe the domain and range.



Both the domain and range are all real numbers except 0.

On Your Own

3. $y = 5x$



4. See Additional Answers.

English Language Learners

Vocabulary

Students may confuse the words *variation* and *variable*. A variable is a symbol, usually a letter, that represents a number that changes. Stress that variation refers to how the variable y varies in relation to the variable x.

Laurie's Notes

Example 2

- **Big Idea:** When y varies directly with x, only one ordered pair is needed to write an equation of the line. This ordered pair can be used to find k, the constant of proportionality.
- Note that the graphs of direct variation equations pass through the origin. So, you know another ordered pair, (0, 0).

? "What ordered pair satisfies the equation you are trying to write?" (12, −6)

- Write the direct variation equation. Substitute for x and y to solve for k.

? "What other ordered pairs satisfy the equation?" Answers will vary. Be sure students understand that for all ordered pairs, $\dfrac{y}{x} = -\dfrac{1}{2}$.

Example 3

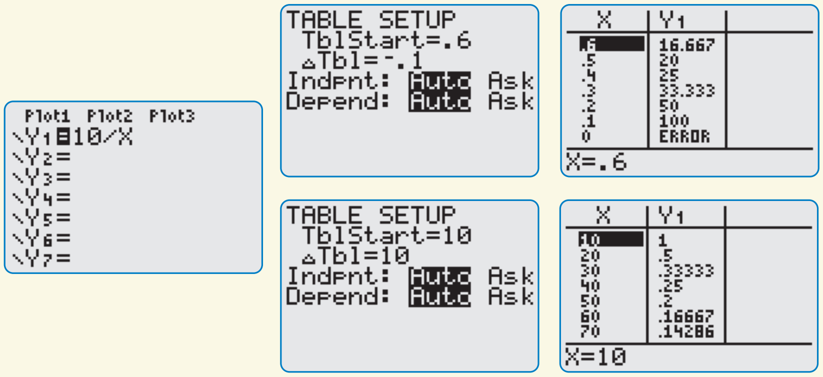
- **Big Idea:** When y varies inversely with x, only one ordered pair is needed to write an equation. This ordered pair can be used to find k, the constant of variation.

? "What ordered pair satisfies the equation you are trying to write?" (2, 5)

- Write the inverse variation equation. Substitute for x and y to solve for k.

? "What other ordered pairs satisfy the equation?" Answers will vary. Be sure students understand that for all ordered pairs, $xy = 10$.

- The table of values in part (b) indicates that the function is undefined when $x = 0$. Division by 0 is undefined.
- **FYI:** The graph is a *hyperbola* with vertical and horizontal *asymptotes*. Asymptotes will be explored more in the next lesson.
- **MP5 Use Appropriate Tools Strategically:** If time permits, explore the function using a graphing calculator. Use a table and small increments of x to explore what happens as x gets closer to 0. Use larger increments to explore what happens as x increases.



On Your Own

- **Neighbor Check:** Have students work independently and then have their neighbors check their work. Have students discuss any discrepancies.

EXAMPLE 2 Writing and Graphing a Direct Variation Equation

Reading

For direct variation equations, you can say "y varies directly with x" or "y is directly proportional to x." For inverse variation equations, you can say "y varies inversely with x" or "y is inversely proportional to x."

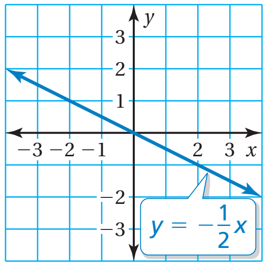
The variable y varies directly with x. When $x = 12$, $y = -6$. Write and graph a direct variation equation that relates x and y.

Find the value of k.

$y = kx$ Write the direct variation equation.

$-6 = k(12)$ Substitute 12 for x and -6 for y.

$-\frac{1}{2} = k$ Divide each side by 12.



So, an equation that relates x and y is $y = -\frac{1}{2}x$.

EXAMPLE 3 Writing and Graphing an Inverse Variation Equation

The variable y varies inversely with x. When $x = 2$, $y = 5$.

a. Write an inverse variation equation that relates x and y.

Find the value of k.

$y = \frac{k}{x}$ Write the inverse variation equation.

$5 = \frac{k}{2}$ Substitute 2 for x and 5 for y.

$10 = k$ Multiply each side by 2.

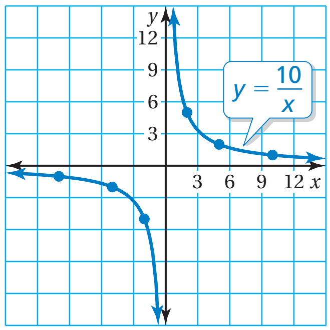
So, an equation that relates x and y is $y = \frac{10}{x}$.

b. Graph the inverse variation equation. Describe the domain and range.

Make a table of values.

x	-10	-5	-2	0	2	5	10
y	-1	-2	-5	undef.	5	2	1

Plot the ordered pairs. Draw a smooth curve through the points in each quadrant. Both the domain and range are all real numbers except 0.



Study Tip

Notice that the equation $y = \frac{10}{x}$ is undefined when $x = 0$. So, there is no point on the graph for $x = 0$.

On Your Own

Now You're Ready
Exercises 7–12

3. The variable y varies directly with x. When $x = 3$, $y = 15$. Write and graph a direct variation equation that relates x and y.

4. The variable y varies inversely with x. When $x = 5$, $y = 4$. Write and graph an inverse variation equation that relates x and y.

EXAMPLE 4 Identifying Inverse Variation

Which situation represents inverse variation?

Ⓐ You buy several movie tickets for $7.50 each.

Ⓑ You earn $0.50 for each pound of aluminum cans you recycle.

Ⓒ The cost of a $600 cabin rental is shared equally by a group of friends.

Ⓓ You download several songs for $0.99 each.

Make a table of values for each situation.

Ⓐ

Number of tickets, x	1	2	3
Total cost, y	7.50	15	22.50

The ratio $\frac{y}{x}$ is constant.

Ⓑ

Number of pounds, x	1	2	3
Total earned, y	0.50	1	1.50

The ratio $\frac{y}{x}$ is constant.

Ⓒ

Number of people, x	1	2	3
Cost per person, y	600	300	200

The product xy is constant.

Ⓓ

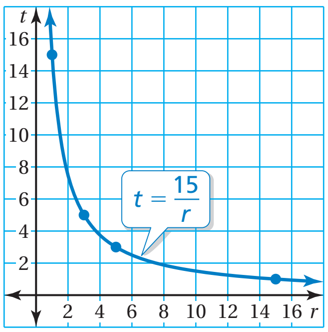
Number of songs, x	1	2	3
Total cost, y	0.99	1.98	2.97

The ratio $\frac{y}{x}$ is constant.

∴ The correct answer is Ⓒ.

EXAMPLE 5 Real-Life Application

You bike 15 miles each morning. Your time t (in hours) to bike 15 miles is given by $t = \frac{15}{r}$, where r is your average speed (in miles per hour). Graph the function. Make a conclusion from the graph.



Because average speed cannot be negative, use only nonnegative values of r.

r	0	1	3	5	15
t	undef.	15	5	3	1

From the graph, you can see that as your average speed increases, the time it takes you to bike 15 miles decreases.

On Your Own

Now You're Ready
Exercises 23–25

5. The cost of a taxi ride is shared equally by several friends. Does this situation represent direct variation or inverse variation? Explain.

6. **WHAT IF?** In Example 5, you bike 12 miles each morning. Write and graph a function that represents your time. Then make a conclusion from the graph.

Laurie's Notes

Example 4

- Ask a volunteer to read each of the four scenarios.
- For each choice, you want students to think about what happens to y as x increases. In three of the four choices, y increases proportionally. Only in the third choice does y decrease.
- Show the checks for constant ratios ($y : x$) and constant products (xy).

Example 5

- **MP2 Reason Abstractly and Quantitatively:** Reason through this problem with students. The faster your speed, the less time it takes to bike 15 miles. The slower your speed, the more time it takes to bike 15 miles.
- **Connection:** When the equation $y = \frac{15}{x}$ is rewritten as $xy = 15$, it should remind students of the distance, speed, and time formula $rt = d$.
- Be sure students understand that the ordered pairs are (time, speed), where time is measured in hours and speed is measured in miles per hour.
- As x (time) increases, y (speed) decreases. As y (speed) increases, x (time) decreases. These ideas could be explored using a graphing calculator, preparing students for the discussion of asymptotes.

? "What does the ordered pair (1.5, 10) mean?" It takes 1.5 hours to bike 15 miles at 10 miles per hour.

On Your Own

- In Question 5, encourage students to check their answer by assigning a value for the total cost, and then observing what happens as you divide it by greater numbers of people.

Closure

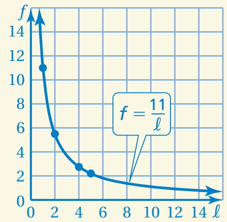
- **Exit Ticket:** Write an equation for the $48 shopping spree at the beginning of the lesson. Let y be the cost per item and let x be the number of items, $xy = 48$ or $y = \frac{48}{x}$.

Extra Example 4

The number of hours h it takes for a block of ice to melt depends on the temperature t of the room. Does this situation represent direct variation or inverse variation? Explain. inverse variation; The number of hours it takes ice to melt is inversely proportional to the temperature of the room.

Extra Example 5

The force f (in pounds) it takes to break a board is given by $f = \frac{11}{\ell}$, where ℓ is the length (in feet) of the board. Graph the function. Make a conclusion from the graph.



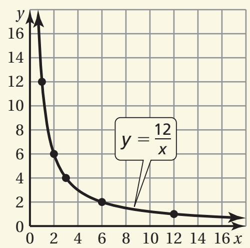
The longer the board, the less force required to break the board.

On Your Own

5. inverse variation; The number of friends is inversely proportional to the cost per person for the taxi ride.

6. The function is $y = \frac{12}{x}$.

   As the amount of time you take to bike 12 miles increases, your average speed decreases.



Vocabulary and Concept Check

1. In direct variation, y is the product of x and the constant k. In inverse variation, y is the quotient of the constant k and x.

2. graph of $g(x)$; It shows direct variation, whereas the rest of the graphs show inverse variation.

Practice and Problem Solving

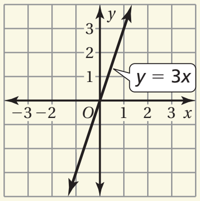
3. direct variation; The ratios $\frac{y}{x}$ are constant.

4. inverse variation; The products xy are constant.

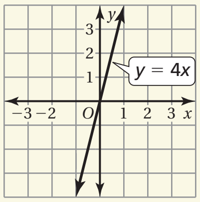
5. direct variation; The equation can be written as $y = kx$.

6. inverse variation; The equation can be written as $y = \frac{k}{x}$.

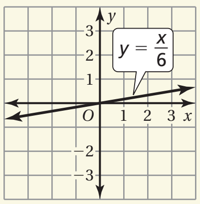
7. $y = 3x$



8. $y = 4x$



9. $y = \frac{1}{6}x$



10–13. See Additional Answers.

Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1, 2, 3–27 odd, 14, 26, 39–42	5, 11, 13, 14, 23, 26
Advanced	1, 2, 4–16 even, 23, 26–38, 39–42	6, 14, 23, 26, 29, 30

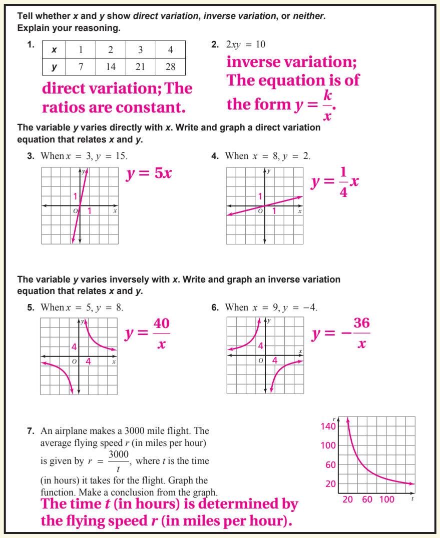
For Your Information

- **Exercises 31–37** A function is *even* if its graph is symmetric with respect to the y-axis. A function is *odd* if its graph is symmetric with respect to the origin.

Common Errors

- **Exercise 3** Students may not believe that x and y show direct variation simply because x and y are increasing by different rates. Remind them to check each product xy and each ratio $\frac{y}{x}$.
- **Exercises 5 and 6** Students may try to identify the type of variation without solving for y. Remind them of the equations $y = kx$ and $y = \frac{k}{x}$.
- **Exercises 7–12** Students may substitute the wrong values for x and y. Tell them to be careful when substituting and that k should still be in the equation after substituting for x and y.

11.1 Record and Practice Journal



Tell whether x and y show *direct variation*, *inverse variation*, or *neither*. Explain your reasoning.

1.

x	1	2	3	4
y	7	14	21	28

direct variation; The ratios are constant.

2. $2xy = 10$

inverse variation; The equation is of the form $y = \frac{k}{x}$.

The variable y varies directly with x. Write and graph a direct variation equation that relates x and y.

3. When $x = 3$, $y = 15$. $y = 5x$

4. When $x = 8$, $y = 2$. $y = \frac{1}{4}x$

The variable y varies inversely with x. Write and graph an inverse variation equation that relates x and y.

5. When $x = 5$, $y = 8$. $y = \frac{40}{x}$

6. When $x = 9$, $y = -4$. $y = -\frac{36}{x}$

7. An airplane makes a 3000 mile flight. The average flying speed r (in miles per hour) is given by $r = \frac{3000}{t}$, where t is the time (in hours) it takes for the flight. Graph the function. Make a conclusion from the graph.

The time t (in hours) is determined by the flying speed r (in miles per hour).

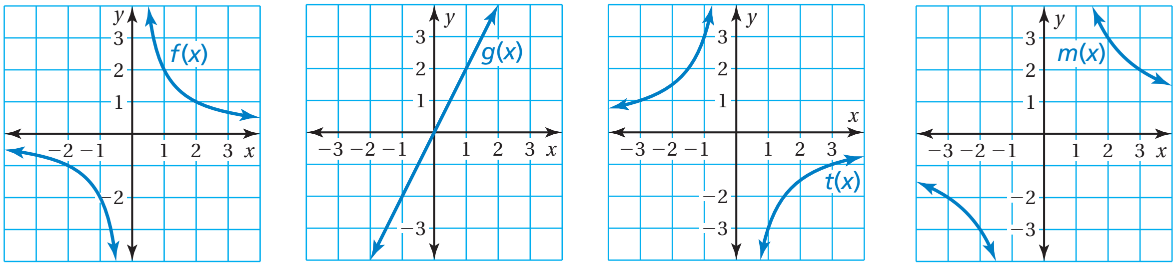
11.1 Exercises

Check It Out
Help with Homework
BigIdeasMath.com

Vocabulary and Concept Check

1. **VOCABULARY** Explain how direct variation equations and inverse variation equations are different.

2. **WHICH ONE DOESN'T BELONG?** Which graph does *not* belong with the other three? Explain your reasoning.



Practice and Problem Solving

Tell whether x and y show *direct variation*, *inverse variation*, or *neither*. Explain your reasoning.

1 3.

x	1	2	3	4
y	2	4	6	8

4.

x	1	2	3	4
y	12	6	4	3

5. $2y = x$

6. $-3xy = 6$

The variable y varies directly with x. Write and graph a direct variation equation that relates x and y.

2 7. When $x = 2$, $y = 6$.

8. When $x = 3$, $y = 12$.

9. When $x = 30$, $y = 5$.

The variable y varies inversely with x. Write and graph an inverse variation equation that relates x and y.

3 10. When $x = 3$, $y = 5$.

11. When $x = 5$, $y = 9$.

12. When $x = 5$, $y = 6$.

13. **VOLUNTEERS** You want to raise \$500 for a charity. You volunteer h hours and raise r dollars each hour. The equation $hr = 500$ represents this situation. Does this represent direct variation, inverse variation, or neither? Explain your reasoning.

14. **ERROR ANALYSIS** The variable y varies inversely with x. When $x = 8$, $y = 5$. Describe and correct the error in writing an inverse variation equation that relates x and y.

$$y = kx$$
$$5 = k(8)$$
$$\frac{5}{8} = k$$
$$\text{So, } y = \frac{5}{8}x.$$

Graph the equation. Describe the domain and range.

15. $y = \dfrac{1}{x}$

16. $\dfrac{y}{x} = -\dfrac{1}{2}$

17. $xy = 9$

18. **REASONING** When y varies directly with x, does x vary directly with y? If so, describe the relationship between the constants of proportionality. Explain your reasoning.

The variable y varies inversely with x. Write an inverse variation equation that relates x and y. Then find the missing value of x or y.

19. When $x = 6$, $y = 2$. Find x when $y = 1$.

20. When $x = 4$, $y = 2$. Find x when $y = \dfrac{1}{2}$.

21. When $x = -2$, $y = -5$. Find y when $x = 4$.

22. When $x = 20$, $y = \dfrac{4}{5}$. Find y when $x = 8$.

Determine whether the situation represents *direct variation* or *inverse variation*. Justify your answer.

4 23. You have enough money to buy 5 hats for \$10 each or 10 hats for \$5 each.

24. Your cousin earns \$50 for mowing 2 lawns or \$75 for mowing 3 lawns.

25. The money the swim team earns from a car wash is divided evenly among the members.

5 26. **RUNNING** You race in a 200-meter dash. Your average speed r (in meters per second) is given by $r = \dfrac{200}{t}$, where t is the time (in seconds) it takes you to finish the race. Graph the function. Make a conclusion from the graph.

27. **VACATION** The amount v of vacation time (in hours) that an employee earns varies directly with the amount t of time (in months) she works. An employee who works 2 months earns 36 hours of vacation time.

   a. Write and graph a direct variation equation that relates v and t.

   b. How many hours of vacation time does the employee earn after working 5 months?

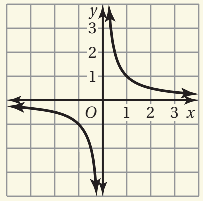
Common Errors

- **Exercises 6, 16 and 17** Students may struggle with multiplying or dividing by x when solving for y. Remind them of the Multiplication and Division Properties of Equality and let them know that these properties apply to variables as well as numbers.
- **Exercises 19–22** Students may substitute the wrong values for x and y. Tell them to be careful when substituting and that k should still be in the equation after substituting for x and y.

Practice and Problem Solving

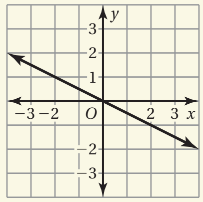
14. An inverse variation equation has the form $y = \frac{k}{x}$; $y = \frac{40}{x}$

15.



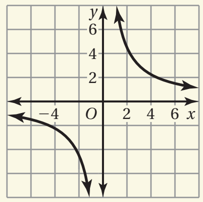
Both the domain and range are all real numbers except 0.

16.



Both the domain and range are all real numbers.

17.



Both the domain and range are all real numbers except 0.

18. yes; The constants are reciprocals of each other.

19. $y = \frac{12}{x}$; $x = 12$

20. $y = \frac{8}{x}$; $x = 16$

21–27. See Additional Answers.

Differentiated Instruction

Connection

The direct variation equation can be written as $\frac{y}{x} = k$. Solving a direct variation problem is similar to setting up and solving a proportion. For Exercise 27(b), $k = \frac{v}{t} = \frac{36}{2}$. So, the proportion is $\frac{36}{2} = \frac{v}{5}$.

Practice and Problem Solving

28. The rate of change of v is not constant whereas the rate of change of d is constant.

29. a. $t = \dfrac{5000}{p}$

b. 200 hours

30. See *Taking Math Deeper.*

31. odd **32.** odd

33. even **34.** neither

35. even **36.** neither

37. a. symmetric with respect to the y-axis

b. symmetric with respect to the origin (reflection in the y-axis followed by a reflection in the x-axis)

38. yes; $f(-x) = -f(x)$ for all direct and inverse variation equations.

Fair Game Review

39–41. See Additional Answers.

42. D

Mini-Assessment

Tell whether *x* and *y* show *direct variation, inverse variation,* or *neither*.

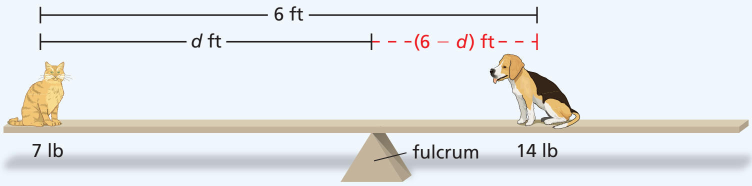
1. $y = x + 9$ neither
2. $y = \frac{1}{6}x$ direct variation
3. $-x - y = 10$ neither
4. $\frac{y}{4} = \frac{3}{x}$ inverse variation
5. One mile is approximately equal to 1.6 kilometers. Determine whether the situation represents direct variation or inverse variation. Write an equation that relates x miles to y kilometers. direct variation; $y = 1.6x$

Taking Math Deeper

Exercise 30

A key to this exercise is realizing how to express the distance from the dog to the fulcrum.

1 Write an expression for the distance from the dog to the fulcrum.



The cat and dog are 6 feet apart and the cat is d feet from the fulcrum. So, the dog is $(6 - d)$ feet from the fulcrum.

2 Let x represent the distance from the animal to the fulcrum and let y represent the weight of the animal. So, the situation can be described by the ordered pairs (distance, weight).

Use each animal to write an ordered pair.

Cat: $(d, 7)$ **Dog:** $(6 - d, 14)$

3 Because the weight varies inversely with the distance, the product xy is equivalent for each pair of points. So, given two points (x_1, y_1) and (x_2, y_2), you know that $x_1y_1 = x_2y_2$. Use this to write and solve an equation.

$$7d = 14(6 - d)$$
$$7d = 84 - 14d$$
$$d = 4$$

So, the cat is 4 feet from the fulcrum and the dog is $6 - 4 = 2$ feet from the fulcrum.

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension Start the next section

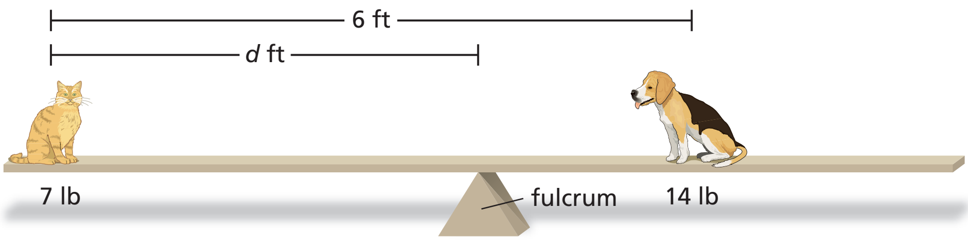
28. **REASONING** Make a table using positive x-values for the inverse variation equation $v = \frac{6}{x}$ and the direct variation equation $d = 6x$. How does the rate of change of v differ from the rate of change of d?

29. **THEATER** A performing arts company is hiring actors as extras for a theater performance. The amount t of performance time (in hours per person) varies inversely with the number p of extras hired. The director estimates that he will need 20 extras performing 250 hours each.

a. Write an inverse variation equation that relates t and p.

b. The director decides to hire 25 extras. How much performance time will each extra receive?

30. **STRUCTURE** To balance the board in the diagram, the distance (in feet) of each animal from the center of the board must vary inversely with its weight (in pounds). What is the distance of each animal from the fulcrum?



A function f is odd if $f(-x) = -f(x)$. A function f is even if $f(-x) = f(x)$. Determine whether the function is *odd*, *even*, or *neither*.

31. $f(x) = x$ **32.** $f(x) = \frac{1}{x}$ **33.** $f(x) = x^2$

34. $f(x) = \sqrt{x}$ **35.** $f(x) = |x|$ **36.** $f(x) = 2^x$

37. **REASONING** Describe the symmetry shown in the graph of (a) an even function and (b) an odd function. Justify your answers.

38. **Precision** Are all direct variation and inverse variation equations odd functions? Explain.

Fair Game Review What you learned in previous grades & lessons

Graph the function. Compare the graph to the graph of $y = x^2$. *(Section 8.1 and Section 8.2)*

39. $y = 3x^2$ **40.** $y = x^2 + 2$ **41.** $y = x^2 - 1$

42. **MULTIPLE CHOICE** What is the solution of the equation $\sqrt{x} - 5 = 4$? *(Section 10.2)*

Ⓐ 1 Ⓑ 3 Ⓒ 9 Ⓓ 81

11.2 Graphing Rational Functions

Essential Question What are the characteristics of the graph of a rational function?

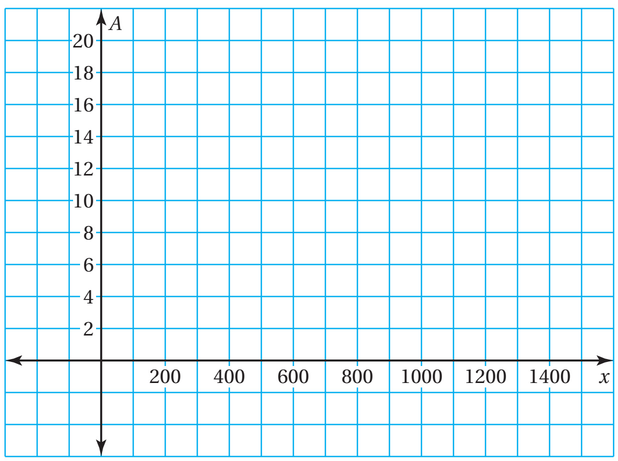
1 ACTIVITY: Graphing a Rational Function

Work with a partner. As a fundraising project, your math club is publishing an optical illusion calendar. The cost of the art, typesetting, and paper is \$850. In addition to this one-time cost, the unit cost of printing each calendar is \$3.25.

a. Let A represent the average cost of each calendar. Write a rational function that gives the average cost of printing x calendars.

$$A = \frac{\boxed{}}{x}$$

b. Make a table showing the average costs for several different production amounts. Then use the table to graph the average cost function.



Common Core

Graphing Rational Functions

In this lesson, you will

- graph rational functions.
- identify asymptotes.
- compare graphs of rational functions.

Learning Standards
A.REI.10
F.BF.4a

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP2 Reason Abstractly and Quantitatively:** Vertical and horizontal asymptotes make sense to students if they have had the opportunity to investigate function values near asymptotes. Mathematically proficient students can reason quantitatively, aided by graphing technology.

Motivate

- Gather 24 like items (counters, paper clips, or wrapped mints).
- When students enter, give one student all 24 items. This will likely be met with comments about why one person got everything.
- Eventually invite a second student to share evenly in the 24 items. Each student now has 12 items.
- Repeat this multiple times, each time discussing what happens when another student joins the recipient group.
- ? "What equation describes the number y of items each person receives when 24 items are divided evenly among x people?" $y = \frac{24}{x}$
- ? "How different would the problem be if I add another item to the 24 original items each time a new person joins the recipient group?" Students may be unsure and that's okay. Today's investigation will help them explore this idea.

Activity Notes

Activity 1

- Discuss with students that there can be one-time costs in a fundraiser, such as purchasing equipment, renting, and licensing.
- **MP1a Make Sense of Problems:** In addition to these fixed costs, there can be costs that vary, such as the cost per item produced. Read through the description with students to be sure they understand the context.
- ? "What is the fixed cost in this problem?" $850 for art, typesetting, and paper
- ? "What is the variable cost in this problem?" cost of printing each calendar
- **MP4 Model with Mathematics:** Make sure students have the correct expression in the numerator of the average cost function.
- The scales on the axes should give students a clue as to what values of x they can use to graph the function.
- **Big Idea:** As more calendars are printed, the fixed cost of $850 is "shared" over more calendars.

Common Core State Standards

A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

F.BF.4a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Previous Learning

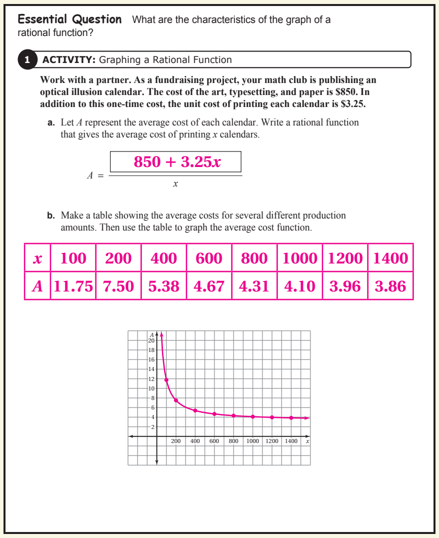
Students should know how to graph inverse variation equations.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.2 Record and Practice Journal



Essential Question What are the characteristics of the graph of a rational function?

1 ACTIVITY: Graphing a Rational Function

Work with a partner. As a fundraising project, your math club is publishing an optical illusion calendar. The cost of the art, typesetting, and paper is $850. In addition to this one-time cost, the unit cost of printing each calendar is $3.25.

a. Let A represent the average cost of each calendar. Write a rational function that gives the average cost of printing x calendars.

$$A = \frac{\boxed{850 + 3.25x}}{x}$$

b. Make a table showing the average costs for several different production amounts. Then use the table to graph the average cost function.

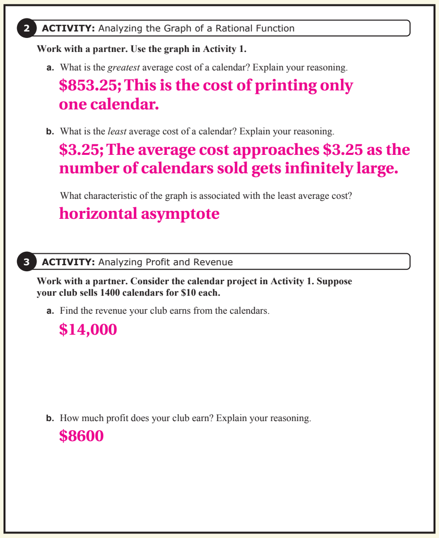
x	100	200	400	600	800	1000	1200	1400
A	11.75	7.50	5.38	4.67	4.31	4.10	3.96	3.86

English Language Learners

Vocabulary

English language learners should see the word *ratio* in the word *rational*. A rational number is a ratio of two integers. A rational function is the ratio of two polynomials.

11.2 Record and Practice Journal



2 ACTIVITY: Analyzing the Graph of a Rational Function

Work with a partner. Use the graph in Activity 1.

a. What is the *greatest* average cost of a calendar? Explain your reasoning.

\$853.25; This is the cost of printing only one calendar.

b. What is the *least* average cost of a calendar? Explain your reasoning.

\$3.25; The average cost approaches \$3.25 as the number of calendars sold gets infinitely large.

What characteristic of the graph is associated with the least average cost?

horizontal asymptote

3 ACTIVITY: Analyzing Profit and Revenue

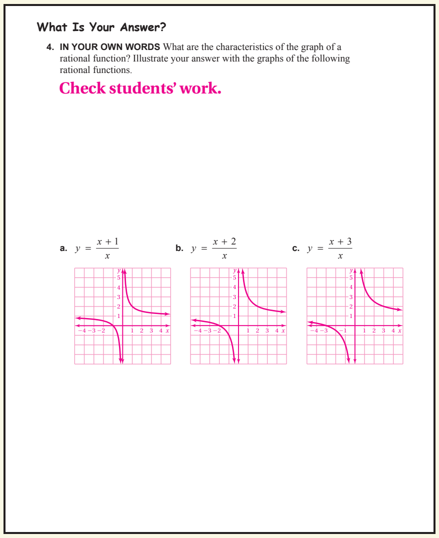
Work with a partner. Consider the calendar project in Activity 1. Suppose your club sells 1400 calendars for \$10 each.

a. Find the revenue your club earns from the calendars.

\$14,000

b. How much profit does your club earn? Explain your reasoning.

\$8600



What Is Your Answer?

4. IN YOUR OWN WORDS What are the characteristics of the graph of a rational function? Illustrate your answer with the graphs of the following rational functions.

Check students' work.

a. $y = \frac{x+1}{x}$ **b.** $y = \frac{x+2}{x}$ **c.** $y = \frac{x+3}{x}$

Laurie's Notes

Activity 2

- Students are now asked to interpret the graph that they made in Activity 1.
- Selling just 1 calendar gives an average cost of \$853.25.
- The average cost continues to decrease as the number of calendars sold increases.
- The average cost cannot go below \$3.25, the unit cost of printing each calendar.
- This can be seen in the graph. As x increases, y approaches 3.25, but never reaches it.
- **Big Idea:** Students may recognize that the function can be written as $A = \frac{850}{x} + 3.25$.

Activity 3

? "How do you calculate profit?" Profit = Revenue − Expenses
- Remind students of the \$850 fixed costs.
- When students have finished, have them share their reasoning.
- **MP5 Use Appropriate Tools Strategically:** Students could create a spreadsheet to explore this problem.

What Is Your Answer?

- If using graphing calculators, students should also explore the functions using tables.

Closure

- **Exit Ticket:** The variable cost increases to \$3.50 per calendar. How does this change your club's profit when selling 1400 calendars for \$10 each? decreases profit to \$8250

2 ACTIVITY: Analyzing the Graph of a Rational Function

Math Practice 3

Justify Conclusions

What information can you use to justify your conclusion?

Work with a partner. Use the graph in Activity 1.

a. What is the *greatest* average cost of a calendar? Explain your reasoning.

b. What is the *least* average cost of a calendar? Explain your reasoning. What characteristic of the graph is associated with the least average cost?

3 ACTIVITY: Analyzing Profit and Revenue

Work with a partner. Consider the calendar project in Activity 1. Suppose your club sells 1400 calendars for $10 each.

a. Find the revenue your club earns from the calendars.

b. How much profit does your club earn? Explain your reasoning.

What Is Your Answer?

4. **IN YOUR OWN WORDS** What are the characteristics of the graph of a rational function? Illustrate your answer with the graphs of the following rational functions.

a. $y = \dfrac{x+1}{x}$ **b.** $y = \dfrac{x+2}{x}$ **c.** $y = \dfrac{x+3}{x}$

Practice Use what you learned about the graphs of rational functions to complete Exercises 4 and 5 on page 555.

11.2 Lesson

Check It Out
Lesson Tutorials
BigIdeasMath.com

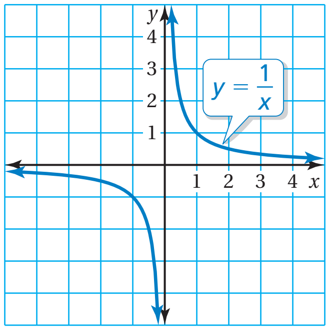
Key Vocabulary
rational function, *p. 552*
excluded value, *p. 552*
asymptote, *p. 553*

The inverse variation equations in Section 11.1 are *rational functions*.

Key Idea

Rational Function

A **rational function** is a function of the form $y = \frac{\text{polynomial}}{\text{polynomial}}$, where the denominator does not equal 0. The most basic rational function is $y = \frac{1}{x}$.



Because division by 0 is undefined, the value of the denominator of a rational function cannot be 0. So, the domain of a rational function *excludes* values that make the denominator 0. These values are called **excluded values** of the rational function.

EXAMPLE 1 Finding the Excluded Value of a Rational Function

Find the excluded value of $y = \frac{2}{x + 5}$.

Find the value of x that makes the denominator 0.

$x + 5 = 0$ Use the denominator to write an equation.

$x = -5$ Subtract 5 from each side.

The excluded value is $x = -5$.

EXAMPLE 2 Graphing a Rational Function

Graph $y = \frac{1}{x - 1}$. Describe the domain and range.

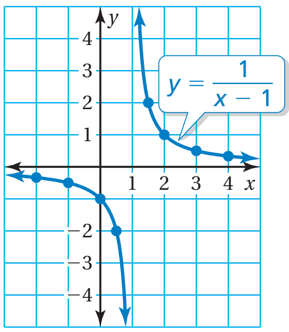
The excluded value is $x = 1$, so choose x-values on either side of 1.

Step 1: Make a table of values.

x	-2	-1	0	0.5	1	1.5	2	3	4
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	undef.	2	1	$\frac{1}{2}$	$\frac{1}{3}$

Step 2: Plot the ordered pairs.

Step 3: Draw a smooth curve through the points on each side of $x = 1$.



The domain is all real numbers except 1 and the range is all real numbers except 0.

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students analyzed a rational function. (MP1a, MP2, MP4, MP5)
- **Today:** Students will graph rational functions.

Motivate

- Play a quick matching game to review previously studied functions and to introduce today's new function.
- Write function types (linear, exponential, quadratic, square root, rational) and sample equations on index cards. Pass them out to students and have them match each equation with a function type.
- Alternatively, you could write them on the board and work as a class. The goal is for students to recognize that there is a new function in the group.

Lesson Notes

Key Idea

- Review polynomials and different sets of numbers, including rational numbers, as a way to introduce rational functions.
- Write the Key Idea, defining a rational function. Point out that inverse variation equations are rational functions.
- You could mention that the graph is called a *hyperbola* and the two parts are called *branches*. Students may recognize the symmetry.
- Discuss the restriction on the denominator.
- Define excluded values of a rational function.

Example 1

- Write the rational function.

? "What value(s) of x will make the value of the denominator 0?" -5

Example 2

? "What value(s) of x will make the value of the denominator 0?" 1

- Because $x = 1$ is excluded from the domain, it is important to use x-values on both sides of 1 in the table of values.
- Point out to students that non-integer values are used near the excluded value. They will appreciate this technique as they graph the function.
- Complete the table of values shown.

? "What happens to the ratio of 1 divided by a positive number when the positive number is increasing?" The ratio approaches zero.

? "What happens to the ratio of 1 divided by a negative number when the negative number is decreasing?" The ratio approaches zero.

- **MP2 Reason Abstractly and Quantitatively:** Similar reasoning can be used to explore what happens to the function values as x gets closer to 1 (from either side). This prepares students for asymptotes.

Goal Today's lesson is graphing **rational functions**.

Technology for the Teacher

Dynamic Classroom

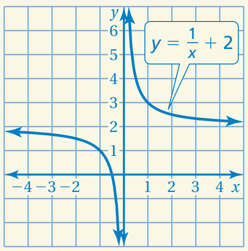
Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Find the excluded value of $y = \dfrac{-3}{2x - 8}$. $x = 4$

Extra Example 2

Graph $y = \dfrac{1}{x} + 2$. Describe the domain and range.



The domain is all real numbers except 0 and the range is all real numbers except 2.

On Your Own

1. $x = 0$
2. $x = 4$
3. $x = -\frac{1}{3}$

4–6. See Additional Answers.

Extra Example 3

Identify the asymptotes of the graph of $y = \frac{1}{x + 6} + 2$. Then describe the domain and range.

$x = -6$, $y = 2$; The domain is all real numbers except -6 and the range is all real numbers except 2.

On Your Own

7. $x = 0$, $y = 1$;
   The domain is all real numbers except 0 and the range is all real numbers except 1.
8. $x = -5$, $y = 0$;
   The domain is all real numbers except -5 and the range is all real numbers except 0.
9. $x = 3$, $y = -2$;
   The domain is all real numbers except 3 and the range is all real numbers except -2.

Differentiated Instruction

Visual

For the graph of $y = \frac{a}{x - h} + k$, the asymptotes, $x = h$ and $y = k$, intersect at (h, k). To graph a rational function, students can locate this point and draw the asymptotes as dashed lines. Then, plot points on each side of the vertical asymptote and connect the points with a smooth curve.

Laurie's Notes

On Your Own

- Guide students in recognizing that there is a connection between the excluded value and the graph.

Discuss

- If you have taken the time to discuss what happens to the value of the function as x gets closer to the excluded value (from either side) or as x goes to infinity (in either direction), the idea of an asymptote will make sense to students.
- Describe asymptotes, referring back to previous examples.

Key Idea

- Write the Key Idea.
- The general form of a rational function may appear to contain many variables to students. Stress that x and y are the variables.
- **MP7 Look for and Make Use of Structure:** Write the general form and refer to the sample graph. The equation in the graph is of this form, where $h = 3$ and $k = 2$. The vertical asymptote is $x = 3$ and the horizontal asymptote is $y = 2$.
- Point out that dotted lines are used to represent the asymptotes.
- **Connection:** Recall that the vertex form of a quadratic function also has h and k in the equation. Students will see that these values shift the graph in the same way.

Example 3

- Write the equation, making note of the general form. Notice the use of color to identify $h = 2$ and $k = -4$.

? "What value of x is excluded from the domain?" 2

- **MP5 Use Appropriate Tools Strategically:** Take time for students to enter the equation in their calculators. Explore the function near $x = 2$ using a table of values.
- **Common Error:** Be sure that students use parentheses to enclose the denominator.
- Discuss the domain and range of the function.
- Students may notice that when using their calculators, there may be a solid vertical line that appears to be an asymptote. In fact, the calculator is just connecting the two parts of the graph.
- Some calculators have an option of "connected" and "dot" modes. The solid vertical line does not appear in dot mode.

On Your Own

- Students should work through the questions on their own, and then use a calculator to check their answers.

On Your Own

Now You're Ready
Exercises 6–17

Find the excluded value of the function.

1. $y = \dfrac{3}{2x}$ **2.** $y = \dfrac{1}{x - 4}$ **3.** $y = \dfrac{8}{3x + 1}$

Graph the function. Describe the domain and range.

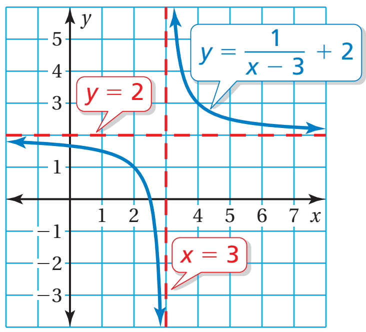
4. $y = -\dfrac{8}{x}$ **5.** $y = \dfrac{1}{x + 2}$ **6.** $y = \dfrac{1}{x} - 1$

The excluded value in Example 2 is $x = 1$. Notice that the graph approaches the vertical line $x = 1$, but never intersects it. The graph also approaches the horizontal line $y = 0$, but never intersects it. These lines are called *asymptotes*. An **asymptote** is a line that a graph approaches, but never intersects.

Key Idea

Asymptotes

The graph of a rational function of the form $y = \dfrac{a}{x - h} + k$, where $a \neq 0$, has a vertical asymptote $x = h$ and a horizontal asymptote $y = k$.



EXAMPLE 3 Identifying Asymptotes

Identify the asymptotes of the graph of $y = \dfrac{1}{x - 2} - 4$. Then describe the domain and range.

Rewrite the function to find the asymptotes.

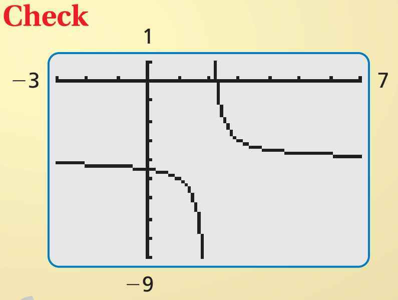
$$y = \frac{1}{x - 2} + (-4)$$

Horizontal Asymptote: $y = -4$

Vertical Asymptote: $x = 2$

The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = -4$. So, the domain of the function is all real numbers except 2 and the range is all real numbers except -4.

Check



On Your Own

Now You're Ready
Exercises 19–24

Identify the asymptotes of the graph of the function. Then describe the domain and range.

7. $y = \dfrac{2}{x} + 1$ **8.** $y = \dfrac{1}{x + 5}$ **9.** $y = \dfrac{8}{x - 3} - 2$

EXAMPLE 4 Comparing Graphs of Rational Functions

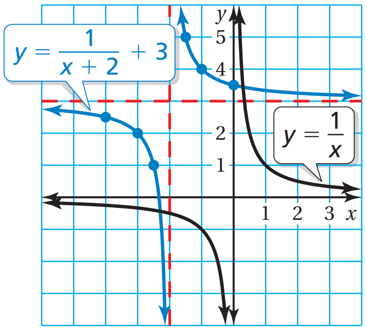
Study Tip

Use the asymptotes to help you draw the ends of the graph.

Graph $y = \dfrac{1}{x+2} + 3$. Compare the graph to the graph of $y = \dfrac{1}{x}$.

Step 1: Make a table of values. The vertical asymptote is $x = -2$, so choose x-values on either side of -2.

x	-4	-3	-2.5	-2	-1.5	-1	0
y	2.5	2	1	undef.	5	4	3.5

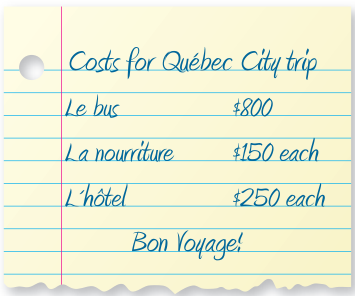


Step 2: Use dashed lines to graph the asymptotes $x = -2$ and $y = 3$. Then plot the ordered pairs.

Step 3: Draw a smooth curve through the points on each side of the vertical asymptote.

The graph of $y = \dfrac{1}{x+2} + 3$ is a translation 3 units up and 2 units left of the graph of $y = \dfrac{1}{x}$.

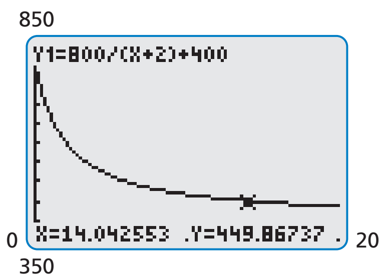
EXAMPLE 5 Real-Life Application



The French club is planning a trip to Québec City. The function $y = \dfrac{800}{x+2} + 400$ represents the cost y (in dollars) per student when x students and 2 chaperones go on the trip. Use a graphing calculator to graph the function. How many students must go on the trip for the cost per student to be about \$450?

Step 1: Use a graphing calculator to graph the function. Because the number of students cannot be negative, use only nonnegative values of x.

Step 2: Use the *trace* feature to find where the value of y is about 450.



About 14 students must go on the trip for the cost per student to be about \$450.

On Your Own

Now You're Ready
Exercises 28–33

Graph the function. Compare the graph to the graph of $y = \dfrac{1}{x}$.

10. $y = \dfrac{1}{x-4}$ **11.** $y = \dfrac{1}{x} - 6$ **12.** $y = -\dfrac{1}{x+3} - 3$

13. **WHAT IF?** In Example 5, how many students must go on the trip for the cost per student to be about \$480?

Laurie's Notes

Example 4

- "How does the graph of $y = (x + 2)^2 + 3$ compare to the graph of $y = x^2$?" translation 2 units to the left and 3 units up
- Say, "Let's see what happens to a rational function with similar values."
- When completing the table of values, students are reviewing rational number operations.
- Describe the translation.

Example 5

- Ask a volunteer to read the problem.
- "What is the domain of this function in context?" positive integers
- **MP5**: Be sure students enter the equation correctly and set a viewing window that is appropriate for the context.
- "What does the shape of the graph indicate about the cost per student?" As more students go on the trip, the cost per student decreases.
- "How do you interpret the horizontal asymptote?" The cost per student cannot be $400 or less.

On Your Own

- **Think-Pair-Share:** Students should read each question independently and then work in pairs to answer the questions. When they have answered the questions, the pair should compare their answers with another group and discuss any discrepancies.

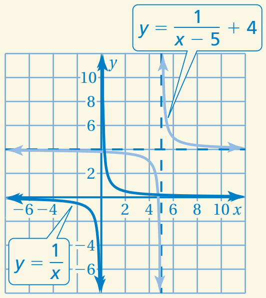
Closure

- **Exit Ticket:** Write an equation of a rational function with a graph that has the vertical asymptote $x = -3$ and the horizontal asymptote $y = 5$.

Sample answer: $y = \dfrac{1}{x + 3} + 5$

Extra Example 4

Graph $y = \dfrac{1}{x - 5} + 4$. Compare the graph to the graph of $y = \dfrac{1}{x}$.



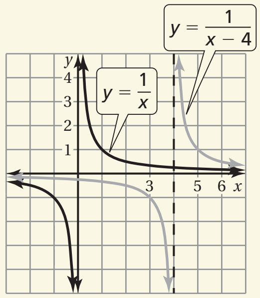
The graph of $y = \dfrac{1}{x - 5} + 4$ is a translation 4 units up and 5 units to the right of the graph of $y = \dfrac{1}{x}$.

Extra Example 5

In Example 5, how many students must go on the trip for the cost per student to be about $425? 30 students

On Your Own

10.



The graph of $y = \dfrac{1}{x - 4}$ is a translation 4 units to the right of the graph of $y = \dfrac{1}{x}$.

11–12. See Additional Answers.

13. about 8 students

Vocabulary and Concept Check

1. no; The denominator is not a polynomial.
2. If the graph of a rational function has a vertical asymptote, it will occur at an excluded value.
3. The graph of a rational function approaches but never intersects the asymptotes.

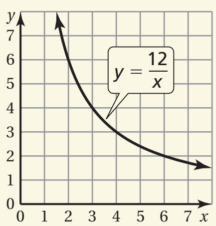
Practice and Problem Solving

4. The graph is two smooth curves. The domain appears to be all real numbers except 0. The range appears to be all real numbers except 10. As x gets closer to 0, the graph approaches the vertical line $x = 0$. As x increases and decreases, the graph approaches the horizontal line $y = 10$.
5. See Additional Answers.
6. $x = 0$
7. $x = -4$
8. $x = -3$
9. $x = 9$
10. $x = 4$
11. $x = -\frac{1}{2}$

12–17. See Additional Answers.

18. **a.** $x = 0$

    b.



The domain is all real numbers greater than 0. The range is all real numbers greater than 0.

Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1–3, 6–18 even, 19–35 odd, 34, 46–49	12, 18, 21, 27, 31, 34
Advanced	1–3, 10, 14, 18, 24–27, 30, 34–45, 46–49	14, 27, 30, 34, 40, 43

For Your Information

- **Exercises 42–44** Students can use a graphing calculator.

Common Errors

- **Exercises 12–17** Students may not plot ordered pairs on either side of the vertical asymptote. They also may not choose x-values that are close to the excluded value. Remind them of this process.
- **Exercise 18** Students may think the domain and range are all real numbers except 0. Negative numbers do not make sense in the context of the problem. The domain is $x > 0$ and the range is $y > 0$. Tell them that real-world problems may impose constraints on the domain and/or range.

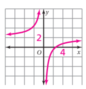
11.2 Record and Practice Journal

Find the excluded value of the function.

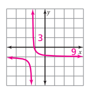
1. $y = \frac{3}{x - 8}$ $\quad$ $x = 8$
2. $y = \frac{1}{2x + 6}$ $\quad$ $x = -3$

Graph the function. Identify the asymptotes of the graph of the function. Describe the domain and range. 3–6. See Additional Answers.

3. $y = -\frac{5}{x} + 2$

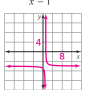


4. $y = \frac{2}{x + 4} - 3$

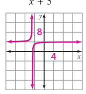


Graph the function. Compare the graph to the graph of $y = \frac{1}{x}$.

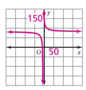
5. $y = \frac{1}{x - 1} - 6$



6. $y = \frac{-1}{x + 5} + 4$



7. A high school football team is taking a trip to an NFL game. Tickets cost \$82 and a charter bus costs \$350. The function $y = \frac{350}{x + 6} + 82$ represents the cost y (in dollars) per player when x players and 6 coaches go on the trip. Graph the function. How many players must go on the trip for the cost per player to be about \$90? **38 players**



11.2 Exercises

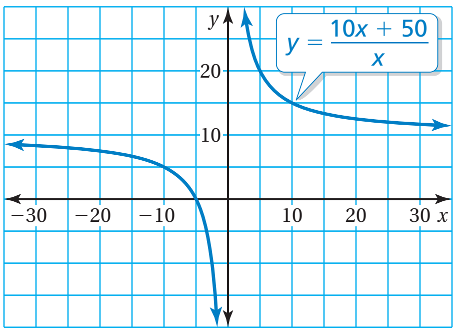
Check It Out
Help with Homework
BigIdeasMath.com

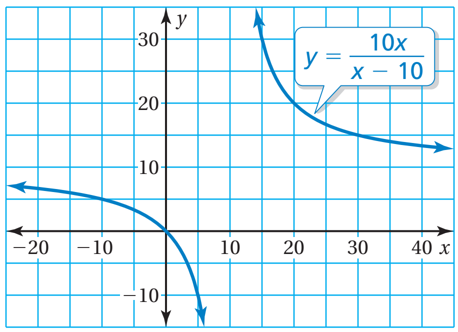
Vocabulary and Concept Check

1. **VOCABULARY** Is $y = \dfrac{1}{\sqrt{x} + 1}$ a rational function? Explain.
2. **VOCABULARY** How is an excluded value related to a vertical asymptote?
3. **WRITING** How can you use asymptotes to help graph a rational function?

Practice and Problem Solving

Describe the characteristics of the graph.

4. 

5. 

Find the excluded value of the function.

1 6. $y = \dfrac{3}{4x}$

7. $y = \dfrac{2}{x + 4}$

8. $y = \dfrac{1}{x + 3}$

9. $y = \dfrac{5}{x - 9}$

10. $y = \dfrac{7}{8 - 2x}$

11. $y = \dfrac{4}{3 + 6x}$

Graph the function. Describe the domain and range.

2 12. $y = \dfrac{5}{x}$

13. $y = \dfrac{2}{5x}$

14. $y = -\dfrac{3}{8x}$

15. $y = \dfrac{1}{x - 3}$

16. $y = \dfrac{4}{x + 1}$

17. $y = \dfrac{1}{4 - 2x}$

18. **HIKING** You hike 12 miles through a national forest to a famous landmark. Your average speed y (in miles per hour) is represented by $y = \dfrac{12}{x}$, where x is the total time (in hours) of the hike.

   a. Find the excluded value of the function.

   b. Graph the function. Describe the domain and range.

Identify the asymptotes of the graph of the function. Then describe the domain and range.

3 **19.** $y = -\frac{6}{x}$

20. $y = \frac{4}{x} + 8$

21. $y = \frac{1}{x-2} + 7$

22. $y = \frac{3}{x+4} - 4$

23. $y = \frac{-2}{x-5} - 2$

24. $y = 10 - \frac{7}{x+9}$

25. ERROR ANALYSIS Describe and correct the error in identifying the asymptotes of the graph of the function.

$y = \frac{3}{x+4} + 5$

The horizontal asymptote is $y = 5$.

The vertical asymptote is $x = 4$.

26. REASONING Describe the domain and range of a rational function of the form $y = \frac{a}{x-h} + k$.

27. OPEN-ENDED Write a rational function whose graph has the vertical asymptote $x = 6$ and the horizontal asymptote $y = -9$.

Graph the function. Compare the graph to the graph of $y = \frac{1}{x}$.

4 **28.** $y = \frac{1}{x} + 2$

29. $y = \frac{1}{x-6}$

30. $y = \frac{1}{x+4} - 2$

31. $y = \frac{1}{x+7} + 3$

32. $y = \frac{-1}{x-1} - 5$

33. $y = 4 - \frac{1}{x+8}$

34. SOFTBALL A softball team buys a new \$250 bat for a softball tournament. The cost of the bat is shared equally by the players on the team. Each player must also pay a \$10 registration fee. The amount y (in dollars) each player pays is represented by $y = \frac{250}{p} + 10$, where p is the number of players on the team. Graph the function. How many players must be on the team for the cost per player to be about \$28?

35. GEOMETRY The formula $h = \frac{2A}{b_1 + b_2}$ gives the height h of a trapezoid, where A is the area and b_1 and b_2 are the base lengths. Suppose $A = 60$ and $b_1 = 8$.

a. Graph the function. Describe the domain and range.

b. Use the graph to find b_2 when $h = 6$.

36. ROAD TRIP The function $t = \frac{280}{r} + 1$ models the total time t (in hours) it takes to drive 280 miles at r miles per hour. The model allows for two half-hour breaks. Graph the function. What does your average speed need to be for the total travel time to be 6 hours?

Common Errors

- **Exercises 19–24** Students may confuse the vertical and horizontal asymptotes. They may also use the wrong sign when identifying asymptotes. Remind them that in the equation $y = \frac{a}{x-h} + k$, the vertical asymptote is $x = h$ and the horizontal asymptote is $y = k$.
- **Exercises 28–33** Students may not identify the horizontal and vertical asymptotes before trying to graph the function. Encourage them to use the asymptotes to help graph the function.
- **Exercise 35** Students may think the domain is all real numbers except -8 and the range is all real numbers except 0. Negative numbers do not make sense in the context of the problem. The domain is $x > 0$ and the range is $0 < y < 15$. Tell them that real-world problems may impose constraints on the domain and/or range.
- **Exercises 37–39** Students may struggle with writing a function for the graph. Tell them to start by identifying the asymptotes. From there, they should be able to work backwards to find an equation of the form $y = \frac{1}{x-h} + k$.

Practice and Problem Solving

19. $x = 0$, $y = 0$; The domain is all real numbers except 0 and the range is all real numbers except 0.

20. $x = 0$, $y = 8$; The domain is all real numbers except 0 and the range is all real numbers except 8.

21. $x = 2$, $y = 7$; The domain is all real numbers except 2 and the range is all real numbers except 7.

22. $x = -4$, $y = -4$; The domain is all real numbers except -4 and the range is all real numbers except -4.

23. $x = 5$, $y = -2$; The domain is all real numbers except 5 and the range is all real numbers except -2.

24. $x = -9$, $y = 10$; The domain is all real numbers except -9 and the range is all real numbers except 10.

25. The wrong sign is used for the vertical asymptote; $x = -4$

26. The domain is all real numbers except h. The range is all real numbers except k.

27. *Sample answer:* $y = \frac{100}{x-6} - 9$

28–36. See Additional Answers.

English Language Learners

Visual Aid

Create a poster of the Key Ideas on pages 552 and 553. Include new vocabulary words and any other important information. Refer to the poster in classroom discussions.

Practice and Problem Solving

37. $y = \dfrac{1}{x-5}$

38. $y = \dfrac{1}{x+3} + 3$

39. $y = 2 - \dfrac{1}{x-1}$

40. If $|a| > 1$, the graph is wider than the graph of $y = \dfrac{1}{x}$.

If $0 < |a| < 1$, the graph is narrower than the graph of $y = \dfrac{1}{x}$.

If $a < 0$, the graph is a reflection in the x-axis of the graph of $y = \dfrac{1}{x}$.

41. about 23°C

42–44. See Additional Answers.

45. See *Taking Math Deeper.*

Fair Game Review

46. nonlinear; This is an inverse variation equation.

47. linear; This is a linear equation in standard form.

48. nonlinear; This is a quadratic equation.

49. D

Mini-Assessment

1. Find the excluded value of $y = \dfrac{5}{3x+9}$. $x = -3$
2. Graph $y = \dfrac{1}{x-8}$. Describe the domain and range.
3. Identify the asymptotes of the graph of $y = \dfrac{1}{x+7} - 5$. Then describe the domain and range.
4. Graph $y = \dfrac{1}{x-5} + 3$. Compare the graph to the graph of $y = \dfrac{1}{x}$.

2–4. See Additional Answers.

Taking Math Deeper

Exercise 45

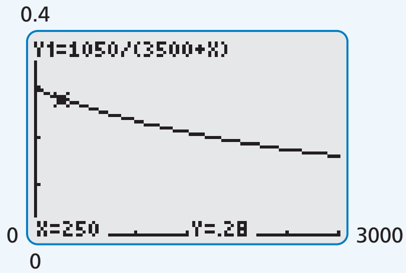
There are a lot of numbers and information in this exercise. Students will need to read carefully and make sense of the problem before continuing.

1 **a.** Write an equation.

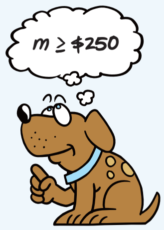
$$r = \frac{\text{expected monthly housing expenses}}{\text{gross monthly income}}$$

$$r = \frac{1050}{3500 + m}$$

2 Graph the equation.



3 **b.** Substitute 0.28 for r and solve algebraically.



$$0.28 = \frac{1050}{3500 + m}$$

$$0.28(3500 + m) = 1050$$

$$980 + 0.28m = 1050$$

$$m = 250$$

The ratio decreases as m increases. So, the ratio is less than or equal to 0.28 when m is greater than or equal to \$250. You can verify this by looking at the graph.

Project

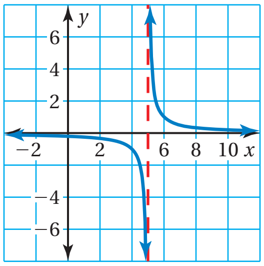
Research mortgage loans. What other criteria are taken into account before a loan is granted?

Reteaching and Enrichment Strategies

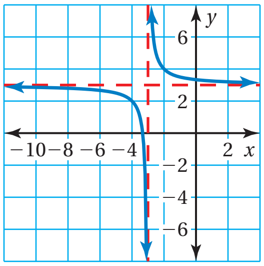
If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension Start the next section

Write a function for the graph.

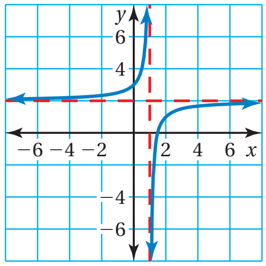
37.



38.



39.



40. REPEATED REASONING Use a graphing calculator to graph the function $y = \dfrac{a}{x-1} + 2$ for several values of a. How does the value of a affect the graph? Consider $a < 0$, $|a| > 1$, and $0 < |a| < 1$ in your answer.

41. THUNDERSTORM The time t (in seconds) it takes for sound to travel 1 kilometer can be represented by $t = \dfrac{1000}{0.6T + 331}$, where T is the temperature in degrees Celsius. Use a graphing calculator to graph the function for $0 \le T \le 100$. During a thunderstorm, lightning strikes 1 kilometer away. You hear the thunder 2.9 seconds later. What is the temperature?

Graph the function. Identify the asymptotes.

42. $y = \dfrac{x}{x+1}$

43. $y = \dfrac{1}{x^2 - 4}$

44. $y = \dfrac{x+1}{x^2 - 1}$

45. **Modeling** To qualify for a mortgage, the ratio r of your expected monthly housing expenses to your gross monthly income cannot be greater than 0.28. Suppose your gross monthly income is \$3500 and you expect to pay \$1050 per month in housing expenses. You also expect to get a raise of m dollars this month.

a. Write and graph an equation that gives r as a function of m.

b. How much must the raise be in order for you to qualify for a mortgage?

Fair Game Review *What you learned in previous grades & lessons*

Does the equation represent a *linear* or *nonlinear* function? Explain. *(Section 5.5)*

46. $3xy = 12$

47. $4x + 2y = 5$

48. $2y - x^2 = 8$

49. MULTIPLE CHOICE Which function models exponential decay? *(Section 6.6)*

Ⓐ $y = -3\left(\frac{1}{2}\right)^x$

Ⓑ $y = -\frac{1}{2}(3)^x$

Ⓒ $y = \frac{1}{2}(3)^x$

Ⓓ $y = 3\left(\frac{1}{2}\right)^x$

Extension 11.2 Inverse of a Function

Check It Out
Lesson Tutorials
BigIdeasMath.com

Key Vocabulary
inverse relation, *p. 558*
inverse function, *p. 559*

Recall that a *relation* pairs inputs with outputs. An **inverse relation** switches the input and output values of the original relation. For example, if a relation contains (a, b), then the inverse relation contains (b, a).

COMMON CORE

Graphing Rational Functions

In this extension, you will
- find inverse functions.

Learning Standards
A.REI.10
F.BF.4a

EXAMPLE 1 Finding Inverse Relations

Find the inverse of each relation.

a. $(-4, 7), (-2, 4), (0, 1), (2, -2), (4, -5)$

Switch the coordinates of each ordered pair.

$(7, -4), (4, -2), (1, 0), (-2, 2), (-5, 4)$

b.

Input	−1	0	1	2	3	4
Output	5	10	15	20	25	30

Switch the inputs and outputs.

Inverse relation:

Input	5	10	15	20	25	30
Output	−1	0	1	2	3	4

Practice

Find the inverse of the relation.

1. $(-5, 8), (-5, 6), (0, 0), (5, 6), (10, 8)$
2. $(-3, -4), (-2, 0), (-1, 4), (0, 8), (1, 12), (2, 16), (3, 20)$
3.

Input	−2	−1	0	1	2
Output	4	1	0	1	4

4.

Input	−2	−1	0	0	1	2
Output	3	4	5	6	7	8

5. **WRITING** How do the domain and range of a relation compare to the domain and range of its inverse relation? Explain.
6. **CRITICAL THINKING** Recall that you can use the Vertical Line Test to determine whether a graph represents a function. What kind of similar test do you think you could use to determine whether a function has an inverse that is also a function? Explain.

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students graphed rational functions. (MP2, MP5, MP7)
- **Today:** Students will find inverses.

Motivate

? "What was the *average daily temperature* yesterday?" Answers will vary. Find this information in advance.

- Write a table on the board listing the date (input) and average daily temperature (output) for a specific location. Feel free to make up the data. Make sure that at least two of the temperatures are the same.

? "Does this table represent a function? Explain." yes; Every date is associated with exactly one average daily temperature.

? "If we switch the inputs and outputs, would it represent a function? Explain." no; An average daily temperature would be associated with more than one date.

Lesson Notes

Discuss

- **MP2 Reason Abstractly and Quantitatively:** Students find inverses by manipulating equations symbolically. Mathematically proficient students are able to reason abstractly, often without a context, to find inverses.
- Students should recall relations, which were introduced earlier in the book.
- Today, the inputs and outputs of a relation are switched, creating an *inverse relation.*

Example 1

- Write the set of ordered pairs in part (a), referring to them as a relation.
- Use color to draw attention to the switching of the coordinates.
- Work through each part as shown.

? "When finding an inverse relation, (x, y) becomes what?" (y, x)

Practice

- **Neighbor Check:** Have students work independently and then have their neighbors check their work. Have students discuss any discrepancies.
- Ask a student to explain the Vertical Line Test. Exercise 5 should suggest a horizontal line test in Exercise 6.

Goal Today's lesson is finding **inverse relations** and **inverse functions**.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Find the inverse of the relation: $(-6, 4)$, $(-3, 2)$, $(0, 0)$, $(3, -2)$, $(6, -4)$.
$(4, -6)$, $(2, -3)$, $(0, 0)$, $(-2, 3)$, $(-4, 6)$

Practice

1. $(8, -5)$, $(6, -5)$, $(0, 0)$, $(6, 5)$, $(8, 10)$
2. $(-4, -3)$, $(0, -2)$, $(4, -1)$, $(8, 0)$, $(12, 1)$, $(16, 2)$, $(20, 3)$
3.

Input	4	1	0	1	4
Output	-2	-1	0	1	2

4.

Input	3	4	5	6	7	8
Output	-2	-1	0	0	1	2

5. The domain of a relation is the range of its inverse relation. The range of a relation is the domain of its inverse relation.
6. Because the domain and range switch, you could use a horizontal line test.

Record and Practice Journal Extension 11.2 Practice

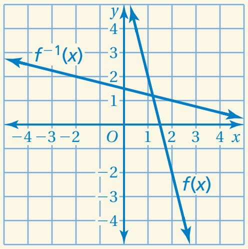
See Additional Answers.

Extra Example 2

Find the inverse of each function. Graph the inverse function.

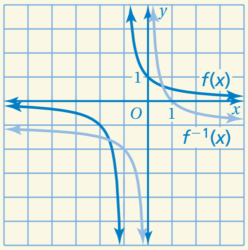
a. $f(x) = 6 - 4x$

$f^{-1}(x) = \frac{6 - x}{4}$



b. $f(x) = \frac{1}{x + 1}$

$f^{-1}(x) = \frac{1}{x} - 1$



Practice

7–12. See Additional Answers.

13. −2

14–15. See Additional Answers.

Mini-Assessment

Find the inverse of the relation.

1. (−2, 5), (−1, 3), (0, 1), (1, 1), (2, 3), (3, 5), (4, 7) (5, −2), (3, −1), (1, 0), (1, 1), (3, 2), (5, 3), (7, 4)

2. (−27, −3), (−8, −2), (−1, −1), (0, 0), (1, 1), (8, 2), (27, 3) (−3, −27), (−2, −8), (−1, −1), (0, 0), (1, 1), (2, 8), (3, 27)

Find the inverse of the function. Graph the inverse function.

3. $f(x) = \frac{1}{5}x - 2$ $f^{-1}(x) = 5x + 10$

See Additional Answers for graph.

4. $f(x) = \frac{1}{x} + 3$ $f^{-1}(x) = \frac{1}{x - 3}$

See Additional Answers for graph.

Laurie's Notes

Discuss

- Refer back to the Motivate, reminding students that the inverse was not a function.
- Introduce the notation, $f^{-1}(x)$. Stress that the −1 is *not* an exponent.

Example 2

- Remind students that $f(x)$ can be replaced by y.
- To find the inverse, switch x and y in the equation. Relate this to switching the coordinates of the ordered pairs in Example 1.
- Solve the equation for y. The last step is to replace y with $f^{-1}(x)$.

? "If the ordered pair (1, −3) satisfies the original function, what ordered pair satisfies the inverse function?" (−3, 1)

- **MP2:** The domain in part (b) is nonnegative, as shown in the graph. So, the *range* of the inverse must be nonnegative. If the domain of the original function had been $x \leq 0$, then the inverse would be $f^{-1}(x) = -\sqrt{x}$.

Practice

- Ask volunteers to show their work at the board for Exercises 7–12.

? "What did you notice about Exercise 11?" Students may say the function did not change. This means that the function is its own inverse, which is always true with inverse variation.

- Exercise 14 helps students recognize the symmetry of functions and their inverses about the line $y = x$.
- **MP1b Persevere in Solving Problems:** You may need to help students work through Exercise 15. Without a lesson on composition of functions, students may have difficulty with the notation. Use a specific example from the lesson to discuss this problem.

Closure

- **Exit Ticket:** Explain the steps you would take to find the inverse of $f(x) = 3x - 4$. Then find the inverse. Replace $f(x)$ with y, switch x and y, solve for y, and then replace y with $f^{-1}(x)$ to obtain $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Reading

The -1 in $f^{-1}(x)$ is not an exponent.

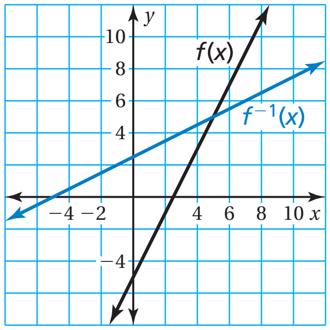
When a relation and its inverse are functions, they are called **inverse functions**. The inverse of a function f is written as $f^{-1}(x)$. To find the inverse of a function represented by an equation, switch x and y and then solve for y.

EXAMPLE 2 Finding Inverse Functions

Find the inverse of each function. Graph the inverse function.

a. $f(x) = 2x - 5$

$y = 2x - 5$	Replace $f(x)$ with y.
$x = 2y - 5$	Switch x and y.
$x + 5 = 2y$	Add 5 to each side.
$\frac{1}{2}x + \frac{5}{2} = y$	Divide each side by 2.
$\frac{1}{2}x + \frac{5}{2} = f^{-1}(x)$	Replace y with $f^{-1}(x)$.

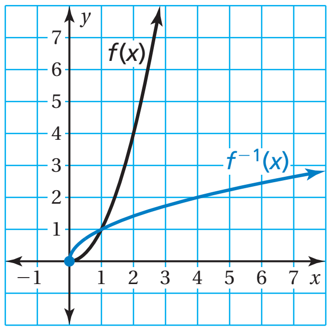


Study Tip

The domain is nonnegative in Example 2b, so the range of the inverse must be nonnegative. This is why you take only the positive square root of each side.

b. $f(x) = x^2$, where $x \geq 0$

$y = x^2$	Replace $f(x)$ with y.
$x = y^2$	Switch x and y.
$\sqrt{x} = y$	Take the positive square root of each side.
$\sqrt{x} = f^{-1}(x)$	Replace y with $f^{-1}(x)$.



Practice

Find the inverse of the function. Graph the inverse function.

7. $f(x) = 3x - 1$

8. $f(x) = -\frac{1}{2}x + 3$

9. $f(x) = 2x^2$, where $x \geq 0$

10. $f(x) = x^2 - 3$, where $x \geq 0$

11. $f(x) = \frac{1}{x}$

12. $f(x) = \frac{1}{x - 2}$

13. **REASONING** Suppose f and f^{-1} are inverse functions and $f(-2) = 5$. What is the value of $f^{-1}(5)$?

14. **REASONING** Draw the line $y = x$ on the graph in each part of Example 2. What do you notice?

15. **LOGIC** Suppose f and g are inverse functions. What do you know about $f(g(x))$ and $g(f(x))$? Explain.

11.3 Simplifying Rational Expressions

Essential Question How can you simplify a rational expression? What are the excluded values of a rational expression?

1 ACTIVITY: Simplifying a Rational Expression

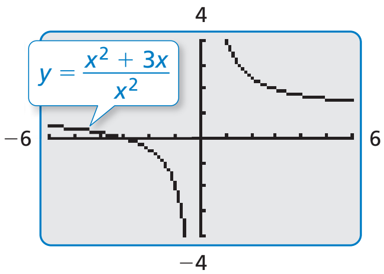
Work with a partner.

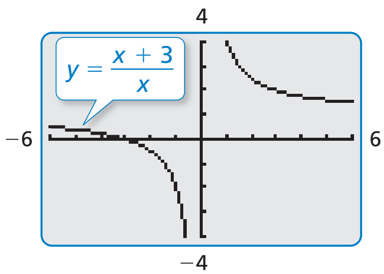
Sample: You can see that the rational expressions

$$\frac{x^2 + 3x}{x^2} \quad \text{and} \quad \frac{x+3}{x}$$

are equivalent by graphing the related functions

$$y = \frac{x^2 + 3x}{x^2} \quad \text{and} \quad y = \frac{x+3}{x}.$$





Both functions have the same graph.

Match each rational expression with its equivalent rational expression. Use a graphing calculator to check your answers.

a. $\frac{x^2 + x}{x^2}$ **b.** $\frac{x^2}{x^2 + x}$ **c.** $\frac{x+1}{x^2 - 1}$ **d.** $\frac{x+1}{x^2 + 2x + 1}$ **e.** $\frac{x^2 + 2x + 1}{x+1}$

A. $\frac{1}{x+1}$ **B.** $x + 1$ **C.** $\frac{x+1}{x}$ **D.** $\frac{1}{x-1}$ **E.** $\frac{x}{x+1}$

Common Core
Radical Expressions
In this lesson, you will
- simplify rational expressions.

Learning Standard
A.SSE.2

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP7 Look for and Make Use of Structure:** Students are simplifying rational expressions and need to remember that common factors can be divided out. Mathematically proficient students understand when an expression is in factored form and can recognize the difference between $3x + 1$ and $3(x + 1)$.

Motivate

- **Puzzle:** If 1 glob equals 8 crabs, 2 crabs equals 5 mils, and 4 mils equals 1 flake, then 1 glob is how many flakes?
- Give students time to discuss this question with a partner.
- One way to solve is shown below.

$$\frac{1\text{ glob}}{8\text{ crabs}} \times \frac{2\text{ crabs}}{5\text{ mils}} \times \frac{4\text{ mils}}{1\text{ flake}} = \frac{1\text{ glob}}{5\text{ flakes}}$$

- In today's lesson, students will work with equivalent expressions.

Activity Notes

Activity 1

? "How can you show that $\dfrac{x^2 + 3x}{x^2}$ is equivalent to $\dfrac{x + 3}{x}$?" Answers will vary. Students may suggest graphing to show that they are equivalent.

- Before graphing, ask students how to factor the numerator of the first expression. It can be factored as $x(x + 3)$. Point out that the numerator and the denominator have a common factor of x.
- Students should now enter the related functions into their graphing calculators.
- **Common Error:** When entering the functions, students may forget to put parentheses around the expression in the numerator.
- **MP1a Make Sense of Problems:** The two graphs will coincide. Students may ask if they are only seeing one of the graphs in this viewing window and the other graph is outside of the viewing window. Checking the table of values will verify that the values are the same for both functions. This helps students make sense of the problem.
- Students should spend time thinking about the rational expressions and how they might simplify an expression in order to find a match.
- **FYI:** Students will learn that the excluded values in the rational function will create a hole in the graph. The holes are sometimes not evident when using a graphing calculator.
- Have students share how they found their matches.

Common Core State Standards

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Previous Learning

Students should know how to graph rational functions.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.3 Record and Practice Journal

Essential Question How can you simplify a rational expression? What are the excluded values of a rational expression?

1 ACTIVITY: Simplifying a Rational Expression

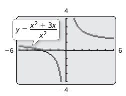
Work with a partner.

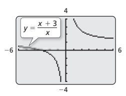
Sample: You can see that the rational expressions

$$\frac{x^2 + 3x}{x^2} \text{ and } \frac{x + 3}{x}$$

are equivalent by graphing the related functions

$$y = \frac{x^2 + 3x}{x^2} \text{ and } \frac{x + 3}{x}.$$





Both functions have the same graph.

Match each rational expression with its equivalent rational expression. Use a graphing calculator to check your answers.

a. $\dfrac{x^2 + x}{x^2}$ C

b. $\dfrac{x^2}{x^2 + x}$ E

c. $\dfrac{x + 1}{x^2 - 1}$ D

d. $\dfrac{x + 1}{x^2 + 2x + 1}$ A

e. $\dfrac{x^2 + 2x + 1}{x + 1}$ B

A. $\dfrac{1}{x + 1}$

B. $x + 1$

C. $\dfrac{x + 1}{x}$

D. $\dfrac{1}{x - 1}$

E. $\dfrac{x}{x + 1}$

Differentiated Instruction

Kinesthetic

A common error when simplifying rational expressions is for students to divide out like terms instead of factors. For instance,

$$\frac{5x}{5(x+2)} = \frac{\cancel{5x}}{\cancel{5x}+10} = \frac{1}{10}$$

Have students write each factor of the original expression on a piece of paper and arrange the pieces as a fraction. Tell students they can only divide out factors when the pieces of paper are identical.

$$\frac{\boxed{5}\,\boxed{x}}{\boxed{5}\,\boxed{(x+2)}} = \frac{\boxed{5}\,\boxed{x}}{\boxed{5}\,\boxed{(x+2)}} = \frac{\boxed{x}}{\boxed{x+2}}$$

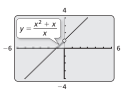
11.3 Record and Practice Journal

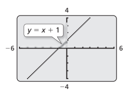
2 ACTIVITY: Finding Excluded Values

Work with a partner. Are the graphs of

$$y = \frac{x^2 + x}{x} \text{ and } y = x + 1$$

exactly the same? Explain your reasoning.





no; The graph of $y = \frac{x^2 + x}{x}$ has a hole at $x = 0$.

3 ACTIVITY: Simplifying and Finding Excluded Values

Work with a partner. Simplify each rational expression, if possible. Then compare the excluded value(s) of the original expression with the excluded value(s) of the simplified expression.

a. $\frac{x^2 + 2x}{x^2}$

$\frac{x+2}{x}$
original: $x = 0$
simplified: $x = 0$

b. $\frac{x^2}{x^2 + 2x}$

$\frac{x}{x+2}$
original: $x = -2, 0$
simplified: $x = -2$

c. $\frac{x^2}{x}$

x
original: $x = 0$
simplified: none

d. $\frac{x^2 + 4x + 4}{x + 2}$

$x + 2$
original: $x = -2$
simplified: none

e. $\frac{x - 2}{x^2 - 4}$

$\frac{1}{x+2}$
original: $x = \pm 2$
simplified: $x = -2$

f. $\frac{1}{x^2 + 1}$

already simplified; no excluded values

What Is Your Answer?

4. **IN YOUR OWN WORDS** How can you simplify a rational expression? What are the excluded values of a rational expression? Include the following rational expressions in your answer.

Factor the numerator and denominator and divide out any common factors; values that make the denominator zero

a. $\frac{x(x + 1)}{x}$

$x + 1$; excluded value: $x = 0$

b. $\frac{x^2 + 3x + 2}{x + 2}$

$x + 1$; excluded value: $x = -2$

c. $\frac{x + 3}{x^2 - 9}$

$\frac{1}{x-3}$; excluded value: $x = \pm 3$

Laurie's Notes

Activity 2

- The hole in the graph on the left should be evident to students.
- ? "Why do you think the graph on the left has a hole at $x = 0$?" Students may realize that $x = 0$ is an excluded value for the rational function.
- ? "Can you factor the numerator in the first function?" yes; $x(x + 1)$
- The numerator and denominator share a common factor of x that can be divided out when simplifying.

Activity 3

- Students may need to factor the numerator and/or denominator to determine if there are common factors that can divide out.
- Explain that excluded values of the original expression are still excluded values in the simplified expression even if it is a common factor that divides out.
- Working through these examples is a good review of factoring and dividing out common factors in a rational expression.
- If time permits, check solutions by graphing the related functions for the original expression and the simplified expression. Remind students to check the table of values, and not just the graphs.

What Is Your Answer?

- **Neighbor Check:** Have students work independently and then have their neighbors check their work. Have students discuss any discrepancies.

Closure

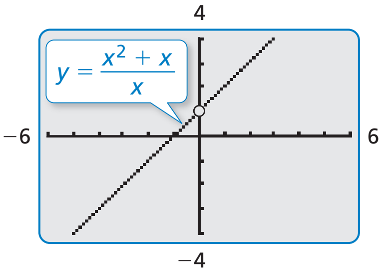
- **Writing Prompt:** Describe how the excluded values of a rational expression can differ from the excluded values of its corresponding simplified expression.

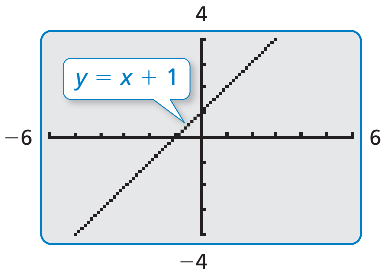
2 ACTIVITY: Finding Excluded Values

Work with a partner. Are the graphs of

$$y = \frac{x^2 + x}{x} \qquad \text{and} \qquad y = x + 1$$

***exactly* the same? Explain your reasoning.**





3 ACTIVITY: Simplifying and Finding Excluded Values

Math Practice 1

Explain the Meaning

What does it mean for a simplified expression to have an excluded value?

Work with a partner. Simplify each rational expression, if possible. Then compare the excluded value(s) of the original expression with the excluded value(s) of the simplified expression.

a. $\dfrac{x^2 + 2x}{x^2}$ **b.** $\dfrac{x^2}{x^2 + 2x}$ **c.** $\dfrac{x^2}{x}$

d. $\dfrac{x^2 + 4x + 4}{x + 2}$ **e.** $\dfrac{x - 2}{x^2 - 4}$ **f.** $\dfrac{1}{x^2 + 1}$

What Is Your Answer?

4. **IN YOUR OWN WORDS** How can you simplify a rational expression? What are the excluded values of a rational expression? Include the following rational expressions in your answer.

a. $\dfrac{x(x + 1)}{x}$ **b.** $\dfrac{x^2 + 3x + 2}{x + 2}$ **c.** $\dfrac{x + 3}{x^2 - 9}$

Practice Use what you learned about simplifying rational expressions to complete Exercises 3–5 on page 564.

Check It Out
Lesson Tutorials
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Key Vocabulary

rational expression, p. 562

simplest form of a rational expression, p. 562

A **rational expression** is an expression that can be written as a fraction whose numerator and denominator are polynomials. Values that make the denominator of the expression zero are *excluded values*.

Key Idea

Simplifying Rational Expressions

Words A rational expression is in **simplest form** when the numerator and denominator have no common factors except 1. To simplify a rational expression, factor the numerator and denominator and *divide out* any common factors.

Algebra Let a, b, and c be polynomials, where $b, c \neq 0$.

$$\frac{ac}{bc} = \frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}$$

Example

$$\frac{2\cancel{(x+1)}}{5\cancel{(x+1)}} = \frac{2}{5};\ x \neq -1$$

Study Tip

You can see why you can *divide out* common factors by rewriting the expression.

$$\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b}$$

EXAMPLE 1 Simplifying Rational Expressions

Simplify each rational expression, if possible. State the excluded value(s).

a. $\dfrac{12}{2x^2} = \dfrac{\cancel{2} \cdot 2 \cdot 3}{\cancel{2} \cdot x \cdot x}$ Divide out the common factor.

$= \dfrac{6}{x^2}$ Simplify.

The excluded value is $x = 0$.

b. $\dfrac{n}{n+8}$

The expression is in simplest form. The excluded value is $n = -8$.

c. $\dfrac{3y^2}{6y(y-7)} = \dfrac{\cancel{3} \cdot \cancel{y} \cdot y}{2 \cdot \cancel{3} \cdot \cancel{y} \cdot (y-7)}$ Divide out the common factors.

$= \dfrac{y}{2(y-7)}$ Simplify.

The excluded values are $y = 0$ and $y = 7$.

Study Tip

Make sure you find excluded values using the *original* expression.

On Your Own

Now You're Ready
Exercises 3–8

Simplify the rational expression, if possible. State the excluded value(s).

1. $\dfrac{5y^3}{2y^2}$ **2.** $\dfrac{8x(x+1)}{12x^2}$ **3.** $\dfrac{m+1}{m(m+3)}$

Multi-Language Glossary at BigIdeasMath.com

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students explored how to simplify rational expressions. (MP1a, MP7)
- **Today:** Students will simplify rational expressions by factoring and dividing out common factors.

Motivate

- Write this problem on the board: $\frac{12}{17} \cdot \frac{51}{77} \cdot \frac{1}{2} \cdot \frac{11}{36} \cdot \frac{21}{13} \cdot \frac{26}{3} = ?$
- Have students work with a partner to find the product.
- Using a calculator, students can see that both the numerator and denominator are 3,675,672. So, the product is 1.

? "Without using a calculator, how can you find this product efficiently?" Divide out common factors.

- To be sure that students remember this process, rewrite the problem as $\frac{12}{17} \cdot \frac{3 \cdot 17}{7 \cdot 11} \cdot \frac{1}{2} \cdot \frac{11}{3 \cdot 12} \cdot \frac{3 \cdot 7}{13} \cdot \frac{13 \cdot 2}{3} = 1$ The numerator and denominator have the same factors that you can divide out.

Lesson Notes

Key Idea

- Write the Key Idea and discuss what it means to simplify a rational expression. Refer to the example above that students just completed.
- **MP6 Attend to Precision:** Do not use the word "cancel" in referring to the common factors. You *divide out* common factors. You can justify *dividing out* by using the Quotient of Powers Property.
- When looking at the example with the binomial factor, note that the excluded value is included in the solution. The original expression is not defined for $x = -1$, so this excluded value must be included in the solution. Students might say, "But there isn't even an x in the answer." While this is true, have students think about the related function $y = \frac{2(x+1)}{5(x+1)}$. The domain excludes $x = -1$.

Example 1

- When simplifying rational expressions, it is helpful to repeat the phrase, "divide out common factors" to remind students that the expression must be in factored form.
- For each part of this example, ask students to list the common factors (if any) of the numerator and denominator. Also ask them to explain the excluded values.
- **Common Error:** In part (b), students may be tempted to divide out the n. The n in the denominator is not a factor; it is being added to 8.

Goal Today's lesson is simplifying **rational expressions**.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Simplify each rational expression, if possible. State the excluded value(s).

a. $\frac{21x}{7x^2}$ $\frac{3}{x}$; $x = 0$

b. $\frac{3}{10 - c}$
The expression is in simplest form; $c = 10$

c. $\frac{2w^3}{8w^2(w+5)}$ $\frac{w}{4(w+5)}$; $w = 0$, $w = -5$

On Your Own

1. $\frac{5y}{2}$; $y = 0$
2. $\frac{2(x+1)}{3x}$; $x = 0$
3. The expression is in simplest form; $m = -3$, $m = 0$

Extra Example 2

Simplify each rational expression, if possible. State the excluded value(s).

a. $\frac{d^2 - 4}{2 - d}$ $-d - 2$; $d = 2$

b. $\frac{s^2 - s - 6}{s^2 - 7s - 18}$ $\frac{s - 3}{s - 9}$; $s = -2$, $s = 9$

Extra Example 3

What is the surface area to volume ratio of a cube-shaped substance with side length $2x$? $\frac{3}{x}$

On Your Own

4. $\frac{2}{7}$; $b = -4$

5. $\frac{1}{2a}$; $a = 0$, $a = 3$

6. $-z - 2$; $z = 8$

7. $\frac{6}{x}$

English Language Learners

Visual

Help English language learners with visual clues when dividing out common factors. Color code the factors to reinforce the mathematical concept. For instance, in Example 2(b) color $c + 4$ red, $c - 3$ blue, and $c - 5$ green.

Laurie's Notes

On Your Own

- **Neighbor Check:** Have students work independently and then have their neighbors check their work. Have students discuss any discrepancies.

Example 2

- **MP7 Look for and Make Use of Structure:** The rational expressions in this example require students to factor the expressions in the numerator and denominator first. Rewriting an expression in an equivalent form is understanding the structure of the expression.

? "In part (a), is the numerator factorable?" Students may not recognize that $1 - z^2$ can be factored using the Difference of Two Squares Pattern.

? "Can we divide out $(1 - z)$ and $(z - 1)$? Explain." no; They are not the same.

- Show how $(1 - z)$ can be written as $-(z - 1)$. The expressions $(1 - z)$ and $-(z - 1)$ are equivalent.
- **FYI:** Some students may recognize that $(1 - z)$ and $-(z - 1)$ are equivalent because -1 has been factored out. Most students prefer to see an explanation off to the side the first few times this technique is used.

Example 3

- Ask for a volunteer to read the problem. Discuss what the phrase "reacts faster with other substances" means. Look at the ratio *surface area* to *volume*. If this ratio increases, then the value of the numerator is increasing faster than the value of the denominator.

? "What value or values of x are excluded?" $x = 0$ and negative values of x; The context of the problem eliminates these values.

On Your Own

- Check Question 6 which requires rewriting a factor.

Closure

- Suppose that one of your friends was absent today. Write an email describing how to simplify $\frac{x^2 - 4}{x + 2}$.

EXAMPLE 2 Simplifying Rational Expressions

Simplify each rational expression, if possible. State the excluded value(s).

a. $\dfrac{1 - z^2}{z - 1} = \dfrac{(1 - z)(1 + z)}{z - 1}$ Difference of Two Squares Pattern

$= \dfrac{-(z - 1)(1 + z)}{z - 1}$ Rewrite $1 - z$ as $-(z - 1)$.

$= \dfrac{-\cancel{(z - 1)}(1 + z)}{\cancel{z - 1}}$ Divide out the common factor.

$= -z - 1$ Simplify.

The excluded value is $z = 1$.

b. $\dfrac{c^2 + c - 12}{c^2 - c - 20} = \dfrac{\cancel{(c + 4)}(c - 3)}{\cancel{(c + 4)}(c - 5)}$ Factor. Divide out the common factor.

$= \dfrac{c - 3}{c - 5}$ Simplify.

The excluded values are $c = -4$ and $c = 5$.

EXAMPLE 3 Real-Life Application

In general, as the surface area to volume ratio of a substance increases, it reacts faster with other substances. Write and simplify this ratio for a block of ice that has the shape shown.

$\dfrac{\text{Surface area}}{\text{Volume}} = \dfrac{2(x^2) + 4(2x^2)}{x(x)(2x)}$ Write an expression.

$= \dfrac{\overset{5}{\cancel{10}}\cancel{x^2}}{\cancel{2}x^{\cancel{3}}}$ Simplify. Divide out the common factors.

$= \dfrac{5}{x}$ Simplify.

On Your Own

Now You're Ready
Exercises 10–15

Simplify the rational expression, if possible. State the excluded value(s).

4. $\dfrac{2b + 8}{7b + 28}$ **5.** $\dfrac{2a - 6}{4a^2 - 12a}$ **6.** $\dfrac{z^2 - 6z - 16}{8 - z}$

7. What is the surface area to volume ratio of a cube-shaped substance with edge length x?

11.3 Exercises

Check It Out
Help with Homework
BigIdeasMath.com

Vocabulary and Concept Check

1. **VOCABULARY** Is $\dfrac{\sqrt{x} - 1}{x + 3}$ a rational expression? Explain.

2. **REASONING** Why is it necessary to state excluded values of a rational expression?

Practice and Problem Solving

Simplify the rational expression, if possible. State the excluded value(s).

1 3. $\dfrac{6}{18x}$ 4. $\dfrac{15y^3}{5y^2}$ 5. $\dfrac{n - 1}{n + 1}$

6. $\dfrac{9w^3}{12w^4}$ 7. $\dfrac{4t^2}{2t(t + 11)}$ 8. $\dfrac{16x^2y}{24xy^3}$

9. **ERROR ANALYSIS** Describe and correct the error in stating the excluded value(s).

✗ $\dfrac{x^3}{x^2(x - 3)} = \dfrac{x}{x - 3}$

The excluded value is $x = 3$.

Simplify the rational expression. State the excluded value(s).

2 10. $\dfrac{3b + 9}{8b + 24}$ 11. $\dfrac{5 - 2z}{2z - 5}$ 12. $\dfrac{6a^2 + 12a}{9a^3 + 18a^2}$

13. $\dfrac{4 - y^2}{y^2 - 3y - 10}$ 14. $\dfrac{n^2 + 5n + 6}{n^2 + 8n + 15}$ 15. $\dfrac{3x^3 - 12x}{6x^3 - 24x^2 + 24x}$

16. **WRITING** Is $\dfrac{(x + 2)(x - 5)}{(x - 2)(5 - x)}$ in simplest form? Explain.

17. **RECYCLING** You hang recycling posters on bulletin boards at your school. Simplify the dimensions of the poster.

Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1, 2, 3–19 odd, 16, 23, 26–28	7, 13, 17, 19, 23
Advanced	1, 2, 4–20 even, 17, 22–25, 26–28	14, 18, 23, 24, 25

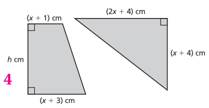
Common Errors

- **Exercises 3–15 and 18–20** Students may not state the correct excluded value(s). Remind them to use the original expression to find the excluded value(s).
- **Exercise 23** Students may not remember the formulas for the volume of a cylinder and the volume of a cone. Remind them of these formulas.
- **Exercise 24** Students may not realize they must add 4 to the area of Sandbox A to make the area equivalent to the area of Sandbox B. Encourage them to identify key phrases before translating the sentence into an equation.

11.3 Record and Practice Journal

Simplify the rational expression, if possible. State the excluded value(s).

1. $\frac{20y^2}{5y}$ — $4y; y = 0$
2. $\frac{18t^2}{12t^4}$ — $\frac{3}{2t^2}; t = 0$
3. $\frac{2d}{d-1}$ — $d = 1$
4. $\frac{27a^2b^2}{36a^3b}$ — $\frac{3b}{4a}; a = 0, b = 0$
5. $\frac{m^3 - 9m^2}{5m^2 - 45m}$ — $\frac{m}{5}; m = 0, 9$
6. $\frac{h^2 + 7}{6 - 2h}$ — $h = 3$
7. $\frac{24x - 6x^2}{x^2 - 3x - 4}$ — $\frac{-6x}{x+1}; x = -1, 4$
8. $\frac{p^2 - 7p - 18}{2p^3 + 10p^2 + 12p}$ — $\frac{p-9}{2p(p-3)}; p = -3, -2, 0$
9. The shapes shown have the same area. Write and simplify an expression for the height h of the trapezoid.



$$h = \frac{(x+4)(2x+4)}{(x+1)+(x+3)} = x + 4$$

Vocabulary and Concept Check

1. no; not a ratio of two polynomials
2. Excluded values make the denominator 0. They must be stated because you cannot divide by 0.

Practice and Problem Solving

3. $\frac{1}{3x}; x = 0$
4. $3y; y = 0$
5. The expression is in simplest form; $n = -1$
6. $\frac{3}{4w}; w = 0$
7. $\frac{2t}{t+11}; t = -11, t = 0$
8. $\frac{2x}{3y^2}; x = 0, y = 0$
9. They did not list all of the excluded values; $x = 0, x = 3$
10. $\frac{3}{8}; b = -3$
11. $-1; z = \frac{5}{2}$
12. $\frac{2}{3a}; a = -2, a = 0$
13. $\frac{2-y}{y-5}; y = -2, y = 5$
14. $\frac{n+2}{n+5}; n = -5, n = -3$
15. $\frac{x+2}{2(x-2)}; x = 0, x = 2$
16. no; Factor -1 out of $(5 - x)$ to get $-(x - 5)$ and then simplify; $-\frac{x+2}{x-2}$
17. $\frac{x}{2}$ by $(x + 3)$

Practice and Problem Solving

18. $\frac{1}{x}$

19. $\frac{3(x+1)}{x(x+3)}$

20. $\frac{2x+3}{x^2+x}$

21. *Sample answer:* $\frac{1}{x^2+8x+15}$

22. The expressions are equivalent for all values of x except -2; Factor $x^2 - 4$ as $(x-2)(x+2)$ and then simplify.

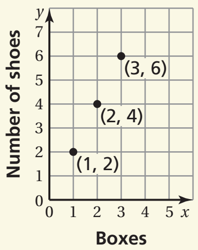
23. $\left(\frac{3}{4}x + 3\right)$ in.

24. $(4x + 4)$ ft

25. See *Taking Math Deeper.*

Fair Game Review

26.



discrete

27. See Additional Answers.

28. C

Mini-Assessment

Simplify the rational expression, if possible. State the excluded value(s).

1. $\frac{2y^3 - 8y^2}{4y^2 - 16y}$ $\frac{y}{2}$; $y = 0$, $y = 4$

2. $\frac{3(m-1)}{5(m+1)}$ The expression is in simplest form; $m = -1$

3. $\frac{z^2 - 15z + 56}{z^2 - 5z - 14}$ $\frac{z-8}{z+2}$; $z = -2$, $z = 7$

Taking Math Deeper

Exercise 25

For this problem, students may have difficulty finding where to begin. Students can look for an entry point to the solution by first looking at the sum $6x^2 + 12x$ and its factors.

1 The expression $6x^2 + 12x$ can be factored as $6x(x + 2)$.

If you add the numerator and denominator of the simplified ratio $\frac{4x+1}{2x-1}$, you get

$$(4x + 1) + (2x - 1) = (4x + 2x) + [1 + (-1)] = 6x.$$

So, you might conclude that the "missing factor" needed is $x + 2$.

2 Now multiply the numerator and denominator of $\frac{4x+1}{2x-1}$ by $x + 2$ because, when simplified, the binomial factors divide out.

$$\frac{4x+1}{2x-1} \cdot \frac{x+2}{x+2} = \frac{(4x+1)(x+2)}{(2x-1)(x+2)}$$

$$= \frac{4x^2 + 9x + 2}{2x^2 + 3x - 2}$$

3 You already know that the rational expression simplifies to $\frac{4x+1}{2x-1}$.

So, check to see that the sum of the two polynomials is $6x^2 + 12x$.

$$(4x^2 + 9x + 2) + (2x^2 + 3x - 2) = (4x^2 + 2x^2) + (9x + 3x) + [2 + (-2)]$$

$$= 6x^2 + 12x \checkmark$$

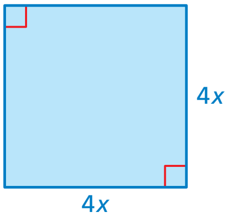
So, $4x^2 + 9x + 2$ and $2x^2 + 3x - 2$ are the two polynomials with simplified ratio $\frac{4x+1}{2x-1}$ and with sum $6x^2 + 12x$.

Reteaching and Enrichment Strategies

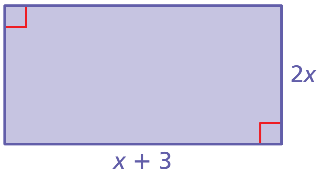
If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension • School-to-Work Start the next section

Write and simplify a rational expression for the ratio of the perimeter of the figure to its area.

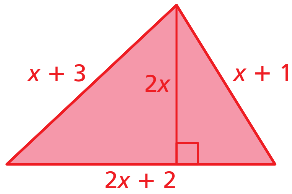
18.



19.

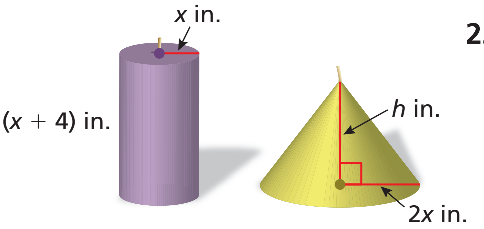


20.



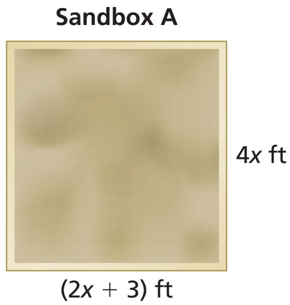
21. **OPEN-ENDED** Write a rational expression whose excluded values are -3 and -5.

22. **WRITING** Is $\dfrac{x^2-4}{x+2}$ equivalent to $x-2$? Justify your answer.



23. **PROBLEM SOLVING** The candles shown have the same volume. Write and simplify an expression for the height of the cone-shaped candle.

24. **SANDBOX** The area of Sandbox B is 4 square feet greater than the area of Sandbox A. Write and simplify an expression for the width w of Sandbox B.



25. Critical Thinking Find two polynomials whose simplified ratio is $\dfrac{4x+1}{2x-1}$ and whose sum is $6x^2 + 12x$. Explain your reasoning.

Fair Game Review What you learned in previous grades & lessons

Graph the function. Is the domain discrete or continuous? *(Section 5.2)*

26.

Input Boxes, x	Output Number of Shoes, y
1	2
2	4
3	6

27.

Input Months, x	Output Height of Plant, y (inches)
1	1.3
2	2.1
3	2.9

28. **MULTIPLE CHOICE** Consider $f(x) = 2x - 4$. What is the value of x so that $f(x) = 8$? *(Section 5.4)*

Ⓐ 2 Ⓑ 4 Ⓒ 6 Ⓓ 7

11 Study Help

Check It Out
Graphic Organizer
BigIdeasMath.com

You can use an **example and non-example chart** to list examples and non-examples of a vocabulary word or term. Here is an example and non-example chart for inverse variation equations.

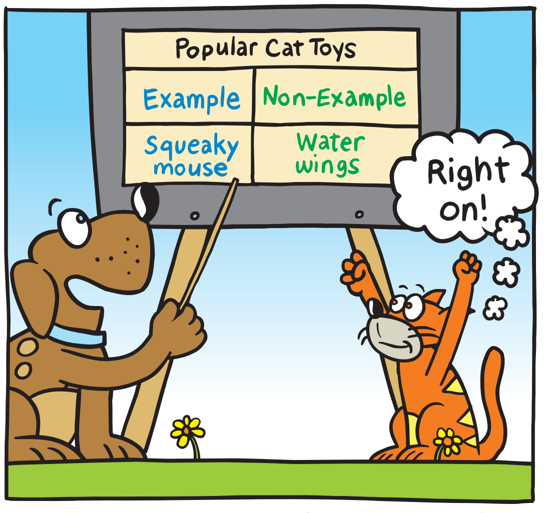
Inverse Variation Equations

Examples	Non-Examples
$y = \frac{2}{x}$	$y = 2x$
$2 = xy$	$2 = \frac{y}{x}$
$x = \frac{2}{y}$	$y = \frac{x}{2}$
$3xy = 6$	$y = 2x + 1$

On Your Own

Make example and non-example charts to help you study these topics.

1. direct variation equations
2. rational functions
3. excluded values
4. asymptotes
5. rational expressions
6. simplest form of a rational expression



"What do you think of my example & non-example chart for popular cat toys?"

After you complete this chapter, make example and non-example charts for the following topics.

7. multiplying and dividing rational expressions
8. least common denominator of rational expressions
9. adding and subtracting rational expressions
10. rational equations

Sample Answers

1.

Direct Variation Equations

Examples	Non-Examples
$y = 3x$	$3 = xy$
$3y = x$	$3xy = 6$
$3 = \frac{y}{x}$	$x = \frac{3}{y}$
$3 = \frac{x}{y}$	$y = \frac{3}{x}$

2.

Rational Functions

Examples	Non-Examples
$y = \frac{1}{x}$	$y = x$
$y = \frac{1}{x+1}$	$y = x + 5$
$y = \frac{x+1}{x-3}$	$y = \frac{\sqrt{x+2}}{x-1}$
$y = \frac{x^2+1}{x^3-2x}$	$y = \frac{1}{\sqrt{x}}$

3.

Excluded Values

Examples	Non-Examples
$x = 0$ for $y = \frac{1}{x}$	$x = 0$ for $y = \frac{x}{x+1}$
$x = 1$ for $y = \frac{1}{x-1}$	$x = 2$ for $y = \frac{x-2}{x+4}$
$x = -1, x = 2$ for $y = \frac{x+3}{(x+1)(x-2)}$	$x = 1, x = -2$ for $y = \frac{x+3}{(x+1)(x-2)}$
$x = 0, x = 2$ for $y = \frac{x+1}{3x(x-2)}$	$x = -3, x = -2$ for $y = \frac{x+1}{3x(x-2)}$

4.

Asymptotes

Examples	Non-Examples
$x = 2, y = 3$ for $y = \frac{1}{x-2} + 3$	$x = -2, y = 0$ for $y = \frac{1}{x} - 2$
$x = 2, y = 0$ for $y = \frac{1}{x-2}$	$x = 0, y = 2$ for $y = \frac{1}{x-2}$
$x = -2, y = -1$ for $y = \frac{5}{x+2} - 1$	$x = 5, y = 1$ for $y = \frac{5}{x+2} - 1$

5–6. Available at *BigIdeasMath.com*.

List of Organizers

Available at *BigIdeasMath.com*

Comparison Chart
Concept Circle
Definition (Idea) and Example Chart
Example and Non-Example Chart
Formula Triangle
Four Square
Information Frame
Information Wheel
Notetaking Organizer
Process Diagram
Summary Triangle
Word Magnet
Y Chart

About this Organizer

An **Example and Non-Example Chart** can be used to list examples and non-examples of a vocabulary word or term. Students write examples of the word or term in the left column and non-examples in the right column. This type of organizer serves as a good tool for assessing students' knowledge of pairs of topics that have subtle but important differences, such as complementary and supplementary angles. Blank example and non-example charts can be included on tests or quizzes for this purpose.

Technology for the Teacher

Editable Graphic Organizer

Answers

1. inverse variation; The products xy are constant.
2. direct variation; The ratios $\frac{y}{x}$ are constant.

3–5. See Additional Answers.

6. $x = 0$
7. $x = \frac{5}{4}$
8. $x = 0$; $y = -5$; The domain is all real numbers except 0 and the range is all real numbers except -5.
9. $x = 0$, $y = 0$; The domain is all real numbers except 0 and the range is all real numbers except 0.

10–12. See Additional Answers.

13. $\frac{1}{2y}$; $y = 0$
14. The expression is in simplest form; $z = -2$
15. $\frac{x-4}{2x+7}$; $x = -2$; $x = -\frac{7}{2}$
16. $2x^2$ by $(2x - 1)$
17. **a.** $c = \frac{400}{n}$

    b. $50

Technology for the Teacher

Online Assessment
Assessment Book
ExamView® Assessment Suite

Alternative Quiz Ideas

100% Quiz	Math Log
Error Notebook	Notebook Quiz
Group Quiz	Partner Quiz
Homework Quiz	Pass the Paper

Homework Quiz

A homework notebook provides an opportunity for teachers to check that students are doing their homework regularly. Students keep their homework in a notebook. They should be told to record the page number, problem number, and copy the problem exactly in their homework notebook. Each day the teacher walks around and visually checks that homework is completed. Periodically, without advance notice, the teacher tells the students to put everything away except their homework notebook.

Questions are from students' homework.

1. What are the answers to Exercises 8 and 10 on page 547?
2. What are the answers to Exercises 23–25 on page 548?
3. What are the answers to Exercises 6 and 10 on page 555?
4. What are the answers to Exercises 19 and 21 on page 556?
5. What are the answers to Exercises 8 and 10 on page 559?
6. What are the answers to Exercises 4, 10, and 14 on page 564?

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Study Help • Practice A and Practice B • Puzzle Time Lesson Tutorials *BigIdeasMath.com*	Resources by Chapter • Enrichment and Extension • School-to-Work Game Closet at *BigIdeasMath.com* Start the next section

11.1–11.3 Quiz

Tell whether x and y show *direct variation*, *inverse variation*, or *neither*. Explain your reasoning. *(Section 11.1)*

1.

x	y
1	-60
2	-30
3	-20
4	-15

2.

x	y
1	6
2	12
3	18
4	24

3.

x	y
1	-2
2	1
3	5
4	10

4. The variable y varies directly with x. When $x = 3$, $y = 15$. Write and graph a direct variation equation that relates x and y. *(Section 11.1)*

5. The variable y varies inversely with x. When $x = 2$, $y = 7$. Write and graph an inverse variation equation that relates x and y. *(Section 11.1)*

Find the excluded value of the function. *(Section 11.2)*

6. $y = \dfrac{2}{5x}$

7. $y = \dfrac{1}{4x - 5}$

Identify the asymptotes of the graph of the function. Then describe the domain and range. *(Section 11.2)*

8. $y = \dfrac{2}{x} - 5$

9. $y = -\dfrac{10}{x}$

10. $y = \dfrac{3}{x + 6} + 9$

Find the inverse of the function. Graph the inverse function. *(Section 11.2)*

11. $f(x) = 2x + 3$

12. $f(x) = x^2 + 1$, where $x \geq 0$

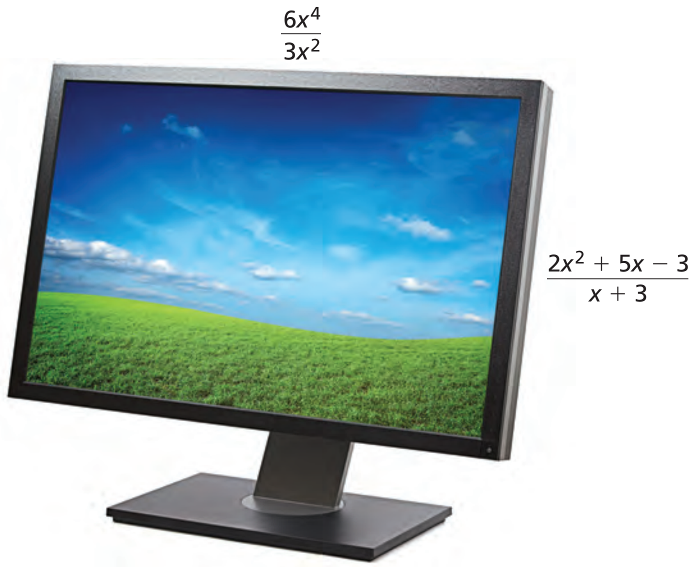
Simplify the rational expression, if possible. State the excluded value(s). *(Section 11.3)*

13. $\dfrac{12y^4}{24y^5}$

14. $\dfrac{2z - 1}{z + 2}$

15. $\dfrac{x^2 - 2x - 8}{2x^2 + 11x + 14}$

16. **DIMENSIONS** Simplify the dimensions of the computer monitor. *(Section 11.3)*



17. **FISHING BOAT** The cost c per person to charter a fishing boat varies inversely with the number n of people fishing. The cost to charter a boat for an entire day is \$400. *(Section 11.1)*

a. Write an inverse variation equation that relates c and n.

b. How much does each person pay when 8 people fish?

11.4 Multiplying and Dividing Rational Expressions

Essential Question How can you multiply and divide rational expressions?

1 ACTIVITY: Matching Quotients and Products

Work with a partner. Match each quotient with a product and then with a simplified expression. Explain your reasoning.

Quotient of Two Rational Expressions	*Product of Two Rational Expressions*	*Simplified Expression*
a. $\frac{2x^2}{5} \div \frac{14x}{10}$	**A.** $\frac{5x^2}{2} \cdot \frac{10}{14x}$	**1.** $\frac{25x}{14}$
b. $\frac{2x^2}{5} \div \frac{10}{14x}$	**B.** $\frac{2x^2}{5} \cdot \frac{10}{14x}$	**2.** $\frac{7x^3}{2}$
c. $\frac{5x^2}{2} \div \frac{14x}{10}$	**C.** $\frac{5x^2}{2} \cdot \frac{14x}{10}$	**3.** $\frac{14x^3}{25}$
d. $\frac{5x^2}{2} \div \frac{10}{14x}$	**D.** $\frac{2x^2}{5} \cdot \frac{14x}{10}$	**4.** $\frac{2x}{7}$

e. $\frac{x^2-1}{x+2} \div \frac{x+1}{x^2-4}$	**E.** $\frac{x^2-1}{x-2} \cdot \frac{x^2-4}{x+1}$	**5.** $x^2 - x - 2$
f. $\frac{x^2-1}{x-2} \div \frac{x+1}{x^2-4}$	**F.** $\frac{x^2-1}{x-2} \cdot \frac{x^2-4}{x-1}$	**6.** $x^2 - 3x + 2$
g. $\frac{x^2-1}{x-2} \div \frac{x-1}{x^2-4}$	**G.** $\frac{x^2-1}{x+2} \cdot \frac{x^2-4}{x-1}$	**7.** $x^2 + x - 2$
h. $\frac{x^2-1}{x+2} \div \frac{x-1}{x^2-4}$	**H.** $\frac{x^2-1}{x+2} \cdot \frac{x^2-4}{x+1}$	**8.** $x^2 + 3x + 2$

i. $\frac{x-1}{2} \div (x-1)$	**I.** $\frac{2}{x^2-1} \cdot (x-1)$	**9.** $\frac{x+1}{2}$
j. $\frac{x^2-1}{2} \div (x-1)$	**J.** $\frac{x^2-1}{2} \cdot \frac{1}{x-1}$	**10.** $\frac{1}{2}$
k. $\frac{x-1}{2} \div \frac{1}{x-1}$	**K.** $\frac{x-1}{2} \cdot \frac{1}{x-1}$	**11.** $\frac{2}{x+1}$
l. $\frac{2}{x^2-1} \div \frac{1}{x-1}$	**L.** $\frac{x-1}{2} \cdot (x-1)$	**12.** $\frac{(x-1)^2}{2}$

COMMON CORE

Rational Expressions

In this lesson, you will
- multiply and divide rational expressions.

Learning Standard
A.SSE.2

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP7 Look for and Make Use of Structure:** Students must use prior skills and techniques to simplify rational expressions. They need to use the structure of an expression to divide out common factors correctly, one from the numerator and one from the denominator.

Motivate

- Write several quotients of fractions on the board.
- ? "How would you solve these problems?" Multiply by the reciprocal of the divisor.
- Ask for volunteers to share their answers.
- The purpose is to review how to divide fractions, a necessary prerequisite for this topic.

Activity Notes

Activity 1

- In this activity, students combine their knowledge of dividing fractions with simplifying rational expressions. This knowledge is generally sufficient for them to work through these problems.
- Repeat the phrase "divide out common factors" as needed while you observe students working each cluster of problems.
- Take time to listen to students explain the reasoning behind their matches. You want to be sure that they arrive at the correct answers for the correct reasons.
- Make sure students do not simply guess towards the end when there are only a few expressions left.

Common Core State Standards

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Previous Learning

Students should know how to simplify rational expressions.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.4 Record and Practice Journal

Essential Question How can you multiply and divide rational expressions?

1 ACTIVITY: Matching Quotients and Products

Work with a partner. Match each quotient with a product and then with a simplified expression. Explain your reasoning.

Quotient of Two Rational Expressions		*Product of Two Rational Expressions*	*Simplified Expression*
a. $\frac{2x^2}{5} \div \frac{14x}{10}$	B, 4	A. $\frac{5x^2}{2} \bullet \frac{10}{14x}$	1. $\frac{25x}{14}$
b. $\frac{2x^2}{5} \div \frac{10}{14x}$	D, 3	B. $\frac{2x^2}{5} \bullet \frac{10}{14x}$	2. $\frac{7x^3}{2}$
c. $\frac{5x^2}{2} \div \frac{14x}{10}$	A, 1	C. $\frac{5x^2}{2} \bullet \frac{14x}{10}$	3. $\frac{14x^3}{25}$
d. $\frac{5x^2}{2} \div \frac{10}{14x}$	C, 2	D. $\frac{2x^2}{5} \bullet \frac{14x}{10}$	4. $\frac{2x}{7}$
e. $\frac{x^2-1}{x+2} \div \frac{x+1}{x^2-4}$	H, 6	E. $\frac{x^2-1}{x-2} \bullet \frac{x^2-4}{x+1}$	5. x^2-x-2
f. $\frac{x^2-1}{x-2} \div \frac{x+1}{x^2-4}$	E, 7	F. $\frac{x^2-1}{x-2} \bullet \frac{x^2-4}{x-1}$	6. x^2-3x+2
g. $\frac{x^2-1}{x-2} \div \frac{x-1}{x^2-4}$	F, 8	G. $\frac{x^2-1}{x+2} \bullet \frac{x^2-4}{x-1}$	7. x^2+x-2
h. $\frac{x^2-1}{x+2} \div \frac{x-1}{x^2-4}$	G, 5	H. $\frac{x^2-1}{x+2} \bullet \frac{x^2-4}{x+1}$	8. x^2+3x+2

English Language Learners

Pair Activity

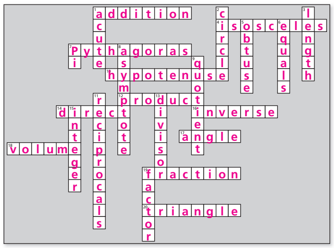
Pair English language learners with English speakers. Provide students with several problems of multiplying and dividing rational expressions. Have one student write the first step and explain what they did to their partner. Then have the other student write the next step and explain. The first person does the next step and the process continues until the problem is completed. Listen for the words *factor* and *divide out.*

11.4 Record and Practice Journal

Quotient of Two Rational Expressions	*Product of Two Rational Expressions*	*Simplified Expression*
i. $\frac{x-1}{2} \div (x-1)$ **K, 10**	I. $\frac{2}{x^2-1} \bullet (x-1)$	9. $\frac{x+1}{2}$
j. $\frac{x^2-1}{2} \div (x-1)$ **J, 9**	J. $\frac{x^2-1}{2} \bullet \frac{1}{x-1}$	10. $\frac{1}{2}$
k. $\frac{x-1}{2} \div \frac{1}{x-1}$ **L, 12**	K. $\frac{x-1}{2} \bullet \frac{1}{x-1}$	11. $\frac{2}{x+1}$
l. $\frac{2}{x^2-1} \div \frac{1}{x-1}$ **I, 11**	L. $\frac{x-1}{2} \bullet (x-1)$	12. $\frac{(x-1)^2}{2}$

2 ACTIVITY: Solving a Math Crossword Puzzle

Work with a partner. Solve the crossword puzzle. Use the clues from the next page.



Across

1. Inverse of subtraction
4. △ or △
7. Greek mathematician
10. Longest side of a right triangle
12. $x(x+1)$
14. $y = kx$
16. $y = \frac{k}{x}$
17.
18. 3 ft^3
19. $\frac{2}{3}$ or $\frac{4}{5}$
20. △

Down

1. 30°
2. C of this is $2\pi r$
3. Dimension of ▭
5. 120°
6. Is the same as
7. About 3.14
8. Graph approaches $x = h$
9. $\frac{x}{x+1}$
11. Two numbers whose product is 1
13. x in $\frac{1}{x}$
15. −1, 0, 1, etc.
19. x in $x(x+1)$

What Is Your Answer?

3. **IN YOUR OWN WORDS** How can you multiply and divide rational expressions? Include the following in your answer.

The product of two rational expressions equals the product of the numerators divided by the product of the denominators. To divide rational expressions, multiply the dividend by the reciprocal of the divisor.

a. $\frac{x+3}{x} \bullet \frac{1}{x+3}$
$= \frac{(x+3) \bullet 1}{x \bullet (x+3)}$
$= \frac{1}{x}$

b. $\frac{x+3}{x} \div \frac{1}{x}$
$= \frac{x+3}{x} \bullet x$
$= \frac{(x+3) \bullet x}{x}$
$= x + 3$

Laurie's Notes

Activity 2

- Students will enjoy working on this crossword puzzle with a partner.
- If students are stuck on a word, they should come back to it later when they may have other clues to help.
- Resist the temptation to give answers or to allow pairs of students to work together.
- **MP1b Persevere in Solving Problems:** Let each pair of students work through the puzzle and persevere through their struggles.

What Is Your Answer?

- **Neighbor Check:** Have students work independently and then have their neighbors check their work. Have students discuss any discrepancies.

Closure

- **Exit Ticket:** Complete the statement.

$$\frac{x^2-1}{?} \bullet \frac{5}{?} = x + 1$$

Sample answer: $\frac{x^2-1}{5} \bullet \frac{5}{x-1} = x + 1$

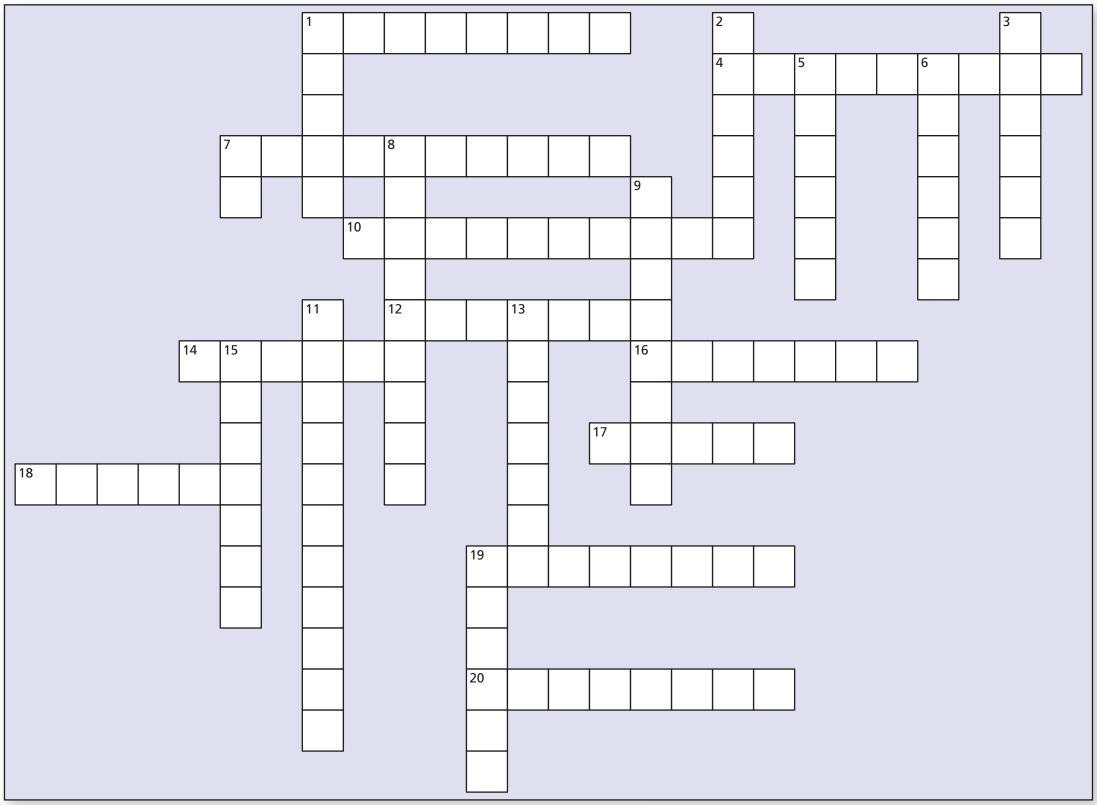
2 ACTIVITY: Solving a Math Crossword Puzzle

Math Practice 3

Use Definitions

How do previously established definitions and results help you to solve this puzzle?

Work with a partner. Solve the crossword puzzle.



Across

1. Inverse of subtraction

4. $\triangle$ or $\triangle$

7. Greek mathematician

10. Longest side of a right triangle

12. $x(x + 1)$

14. $y = kx$

16. $y = \frac{k}{x}$

17. $\angle$

18. 3 ft^3

19. $\frac{2}{3}$ or $\frac{4}{5}$

20. $\triangle$

Down

1. 30°

2. C of this is $2\pi r$

3. Dimension of ▭

5. 120°

6. Is the same as

7. About 3.14

8. Graph approaches $x = h$

9. $\frac{x}{x + 1}$

11. Two numbers whose product is 1

13. x in $\frac{1}{x}$

15. $-1, 0, 1$, etc.

19. x in $x(x + 1)$

What Is Your Answer?

3. IN YOUR OWN WORDS How can you multiply and divide rational expressions? Include the following in your answer.

a. $\frac{x + 3}{x} \cdot \frac{1}{x + 3}$

b. $\frac{x + 3}{x} \div \frac{1}{x}$

Practice Use what you learned about multiplying and dividing rational expressions to complete Exercises 4 and 10 on page 572.

11.4 Lesson

Check It Out
Lesson Tutorials
BigIdeasMath.com

You can use the same rules that you used for multiplying and dividing fractions to multiply and divide rational expressions.

Key Idea

Multiplying and Dividing Rational Expressions

Let a, b, c, and d be polynomials.

Multiplying: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, where $b, d \neq 0$

Dividing: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$, where $b, c, d \neq 0$

EXAMPLE 1 Multiplying Rational Expressions

Find each product.

a. $\frac{5}{2x^3} \cdot \frac{4x^3}{x+1} = \frac{5 \cdot 4x^3}{2x^3(x+1)}$ Multiply numerators and denominators.

$= \frac{5 \cdot \cancel{4x^3}^{2}}{\cancel{2x^3}(x+1)}$ Divide out the common factors.

$= \frac{10}{x+1}$ Simplify.

b. $\frac{h}{h+2} \cdot \frac{h^2+5h+6}{h^2}$

$= \frac{h}{h+2} \cdot \frac{(h+3)(h+2)}{h^2}$ Factor $h^2 + 5h + 6$.

$= \frac{h(h+3)(h+2)}{h^2(h+2)}$ Multiply numerators and denominators.

$= \frac{\cancel{h}(h+3)\cancel{(h+2)}}{\cancel{h^2}(\cancel{h+2})}$ Divide out the common factors.

$= \frac{h+3}{h}$ Simplify.

Remember

Remember that expressions may have excluded values. In Example 1a, the excluded values are $x = -1$ and $x = 0$.

On Your Own

Now You're Ready
Exercises 4–9

Find the product.

1. $\frac{8y^2}{y-5} \cdot \frac{3}{4y}$

2. $\frac{16}{8-c} \cdot (c-8)$

3. $\frac{2z-4}{6} \cdot \frac{3}{z^2-7z+10}$

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students recognized equivalent expressions. (MP1b, MP7)
- **Today:** Students will multiply and divide rational expressions.

Motivate

- Write the following problems on the board.

$$\frac{2}{3} \div \frac{2}{3} \bullet \frac{2}{3} \div \frac{2}{3} \bullet \frac{2}{3} = ? \qquad \frac{2}{3} \div \frac{2}{3} \bullet \frac{2}{3} \div \frac{2}{3} \bullet \frac{2}{3} \div \frac{2}{3} = ?$$

? "What are the answers?" $\frac{2}{3}$; 1

? "If this pattern were continued by adding 5 more terms (11 total fractions), what is the answer? Explain." $\frac{2}{3}$; When the number of terms is even, the answer is 1. When the number of terms is odd, the answer is $\frac{2}{3}$.

- Explain that today's lesson involves multiplying and diving rational expressions.

Lesson Notes

Key Idea

- Write the Key Idea.
- **Big Idea:** The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing fractions. Common factors can be divided out, and the denominator cannot be 0.

Example 1

- Write the problem in part (a). Encourage students to try factoring first and looking for common factors, rather than immediately multiplying.
- Students may need to be reminded of the properties of exponents.
- Write the problem in part (b). If students do not recognize any common factors, put parentheses around the $(h + 2)$ term. It may help them recognize that the trinomial can be factored.

? "What are the common factors?" the $(h + 2)$ terms and an h term

- Discuss the Remember box.

On Your Own

- **Teaching Tip:** Some students try to perform all of the steps mentally and may not even write the original problem. Encourage students to write each step.
- Ask volunteers to share their work and explanations with the class.
- In Question 2, students need to rewrite $(8 - c)$ as $-(c - 8)$.

Goal Today's lesson is multiplying and dividing rational expressions.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Find each product.

a. $\frac{2n-1}{4n^5} \bullet \frac{2n^3}{3}$ $\frac{2n-1}{6n^2}$

b. $\frac{k-3}{k^2} \bullet \frac{k+2}{k^2-k-6}$ $\frac{1}{k^2}$

On Your Own

1. $\frac{6y}{y-5}$
2. -16
3. $\frac{1}{z-5}$

Laurie's Notes

Extra Example 2

Find the quotient $\frac{d+6}{d^2} \div \frac{2d+12}{d^2}$. $\frac{1}{2}$

Extra Example 3

Find the quotient
$\frac{a^2-16}{a+3} \div (a^2+7a+12)$. $\frac{a-4}{(a+3)^2}$

On Your Own

4. $2t$
5. $\frac{g-1}{g}$
6. $\frac{1}{(d-1)^2}$

Differentiated Instruction

Auditory

Guide your students in making the following list of steps used in dividing rational expressions.

1. Multiply by the reciprocal.
2. Factor each numerator and denominator.
3. Multiply the numerators and the denominators.
4. Divide out common factors.
5. Simplify.

Organize students into small groups. Assign each group an expression in the form of $\frac{a}{b} \div \frac{c}{d}$. Have each group simplify the expression and present their solution to the class. Students should read the steps from the list aloud as they explain their solution.

Example 2

? "What is the basic procedure used to divide rational expressions?" Multiply by the reciprocal of the divisor.

- Write the original problem and solve as shown.
- **MP7 Look for and Make Use of Structure:** You may want to ask students how they could do this problem mentally. The two rational expressions have the same denominator, so you can solve by simply finding the quotient of the numerators.

Example 3

- Rewriting the divisor as a fraction will make students less likely to make a mistake when dividing.
- Write the equivalent multiplication problem.

? "Can the p^2-terms be divided out? Explain." No, the p^2-terms are not factors. These polynomials should be factored.

- Continue to work through the problem as shown.

On Your Own

- **Think-Pair-Share:** Students should read each question independently and then work in pairs to answer the questions. When they have answered the questions, the pair should compare their answers with another group and discuss any discrepancies.

Closure

- **Exit Ticket:** Find the quotient: $\frac{x^2-x-2}{5x} \div \frac{x-2}{x^2}$. $\frac{x(x+1)}{5}$

EXAMPLE 2 Dividing Rational Expressions

Which expression is equivalent to $\dfrac{8}{w-4} \div \dfrac{w}{w-4}$ when $w \neq 4$?

Ⓐ $\dfrac{8}{w}$ Ⓑ $\dfrac{w}{8}$ Ⓒ $\dfrac{8w}{(w-4)^2}$ Ⓓ $\dfrac{8w}{w^2-8w+16}$

$$\frac{8}{w-4} \div \frac{w}{w-4} = \frac{8}{w-4} \cdot \frac{w-4}{w}$$ Multiply by the reciprocal.

$$= \frac{8(w-4)}{w(w-4)}$$ Multiply numerators and denominators.

$$= \frac{8\cancel{(w-4)}}{w\cancel{(w-4)}}$$ Divide out the common factor.

$$= \frac{8}{w}$$ Simplify.

∴ The correct answer is Ⓐ.

EXAMPLE 3 Dividing Rational Expressions

Find the quotient $\dfrac{p^2-p-6}{p+1} \div (p^2-4)$.

$$\frac{p^2-p-6}{p+1} \div \frac{p^2-4}{1}$$ Write $p^2 - 4$ as a fraction.

$$= \frac{p^2-p-6}{p+1} \cdot \frac{1}{p^2-4}$$ Multiply by the reciprocal.

$$= \frac{(p-3)(p+2)}{p+1} \cdot \frac{1}{(p-2)(p+2)}$$ Factor.

$$= \frac{(p-3)(p+2)}{(p+1)(p-2)(p+2)}$$ Multiply numerators and denominators.

$$= \frac{(p-3)\cancel{(p+2)}}{(p+1)(p-2)\cancel{(p+2)}}$$ Divide out the common factor.

$$= \frac{p-3}{(p+1)(p-2)}$$ Simplify.

On Your Own

Now You're Ready
Exercises 10–15

Find the quotient.

4. $\dfrac{t-2}{2t} \div \dfrac{t-2}{4t^2}$ **5.** $(g+1) \div \dfrac{g^2+g}{g-1}$ **6.** $\dfrac{d+5}{d-1} \div (d^2+4d-5)$

11.4 Exercises

Check It Out
Help with Homework
BigIdeasMath.com

Vocabulary and Concept Check

1. **WRITING** Describe how to multiply rational expressions.

2. **WRITING** Describe how to divide rational expressions.

3. **NUMBER SENSE** Consider the expressions $\frac{x}{x-2}$ and $\frac{x+1}{x}$. For what value(s) is the product of the expressions undefined? For what value(s) is the quotient of the expressions undefined?

Practice and Problem Solving

Find the product.

1 4. $\frac{5}{3c^2} \cdot \frac{c^5}{15(c-2)}$

5. $\frac{n+3}{8n^6} \cdot \frac{4n^2}{7}$

6. $(d^2-d) \cdot \frac{14}{1-d}$

7. $\frac{x+4}{6x} \cdot \frac{x^2}{x^2-x-20}$

8. $\frac{k^2-8k+15}{5k^3} \cdot \frac{3k}{k-5}$

9. $\frac{-r-6}{2r^2+8r} \cdot \frac{4r^2+16r}{r^2-36}$

Find the quotient.

2 3 10. $\frac{2h}{h+8} \div \frac{16}{h+8}$

11. $\frac{t-5}{9t} \div \frac{t-5}{6t^2}$

12. $\frac{y+7}{7y} \div \frac{3y^2+21y}{14y-5}$

13. $\frac{p^2-16}{p-3} \div (p-4)$

14. $\frac{g^2-4g-21}{4g^2+12g} \div (g-7)$

15. $\frac{3z-27}{z-6} \div (z^2-15z+54)$

ERROR ANALYSIS **Describe and correct the error in finding the quotient.**

16. ✗

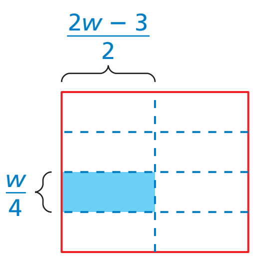
$$\frac{6w}{w+1} \div \frac{3w}{w+1} = \frac{6w}{w+1} \cdot \frac{3w}{w+1} = \frac{6w \cdot 3w}{(w+1)(w+1)} = \frac{18w^2}{(w+1)^2}$$

17. ✗

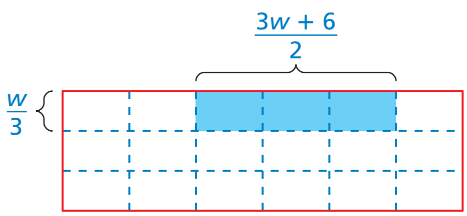
$$\frac{v-2}{4v} \div \frac{v-2}{6v^2} = \frac{4v}{v-2} \cdot \frac{v-2}{6v^2} = \frac{4v(v-2)}{6v^2(v-2)} = \frac{2}{3v}$$

Find the total area of the red rectangle in terms of *w*.

18.



19.



Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1–3, 4–18 even, 20, 21, 23, 24, 27–30	4, 14, 16, 18, 23
Advanced	1–3, 5–21 odd, 22–26, 27–30	9, 17, 19, 23, 26

For Your Information

- **Exercise 26** Students may not remember the order of operations. If students are confused, tell them to think of the problem as $\left(\frac{8x^3}{x+1} \div \frac{2x-2}{3x}\right) \div \frac{16x^2}{x+1}$. The order of operations apply to rational expressions as well as numbers.

Common Errors

- **Exercises 4–15** Students may not factor completely before simplifying the rational expression. Remind students to factor completely in order to divide out common factors so that the rational expression is in simplest form.
- **Exercises 10–15** Students may not rewrite the division as multiplication. Remind students that when they divide rational expressions, they need to multiply the dividend by the reciprocal of the divisor.
- **Exercises 10–15** Students may multiply the divisor by the reciprocal of the dividend. Remind students that when they divide rational expressions, they need to multiply the dividend by the reciprocal of the divisor.

11.4 Record and Practice Journal

Find the product.

1. $\frac{8k^2}{7} \bullet \frac{k-5}{2k^4}$ $\quad \frac{4(k-5)}{7k^2}$

2. $\frac{8}{w+6} \bullet \frac{3w^2+18w}{2}$ $\quad 12w$

3. $\frac{2b^3}{b^2-2b-15} \bullet \frac{b-5}{6b}$ $\quad \frac{b^2}{3b+9}$

4. $\frac{4n^2-8n}{n^2+9n+14} \bullet \frac{n^2-4}{4n}$ $\quad \frac{(n-2)^2}{n+7}$

Find the quotient.

5. $\frac{g+3}{4g^2} \div \frac{g+3}{12g}$ $\quad \frac{3}{g}$

6. $\frac{y}{-5y^2-20y} \div \frac{y^2}{y^2+2y-8}$ $\quad \frac{y-2}{-5y^2}$

7. $\frac{a^2-64}{a} \div (3a^3+24a^2)$ $\quad \frac{a-8}{3a^3}$

8. $\frac{r^2-7r}{3} \div (r^2+r-56)$ $\quad \frac{r}{3r+24}$

9. Two distinct prairie dog populations, P_1 and P_2 can be modeled by $P_1 = \frac{100x^2}{x+1}$ and $P_2 = \frac{100x^2}{x+3}$, where x is the number of years since 2000.

   a. Write a function that models the ratio of Population 1 to Population 2, that is $\frac{P_1}{P_2}$. $\quad \frac{x+3}{x+1}$

   b. Find the ratio of Population 1 to Population 2 in 2004. $\quad \frac{7}{5}$

Vocabulary and Concept Check

1. (1) Factor the numerators and denominators. (2) Multiply the numerators and denominators. (3) Divide out common factors. (4) Simplify.

2. (1) Rewrite the quotient as the product of the dividend and the reciprocal of the divisor. (2) Factor the numerators and denominators. (3) Multiply the numerators and denominators. (4) Divide out common factors. (5) Simplify.

3. $x = 0$, $x = 2$; $x = -1$, $x = 0$, $x = 2$

Practice and Problem Solving

4. $\frac{c^3}{9(c-2)}$
5. $\frac{n+3}{14n^4}$
6. $-14d$
7. $\frac{x}{6(x-5)}$
8. $\frac{3(k-3)}{5k^2}$
9. $\frac{-2}{r-6}$
10. $\frac{h}{8}$
11. $\frac{2t}{3}$
12. $\frac{14y-5}{21y^2}$
13. $\frac{p+4}{p-3}$
14. $\frac{1}{4g}$
15. $\frac{3}{(z-6)^2}$
16. See Additional Answers.
17. To multiply rational expressions, you multiply by the reciprocal of the divisor, not the dividend;

$$\frac{v-2}{4v} \div \frac{v-2}{6v^2} = \frac{v-2}{4v} \bullet \frac{6v^2}{v-2}$$
$$= \frac{6v^2(v-2)}{4v(v-2)} = \frac{3v}{2}$$

18. $w(2w-3)$
19. $3w(w+2)$

Practice and Problem Solving

20. $\dfrac{b+4}{2b+1}$

21. -1

22. $x = -2, x = -1, x = 3$

23. See *Taking Math Deeper.*

24. a. Their graphs coincide.

    b. The y_1 column will display ERROR for x values that are excluded.

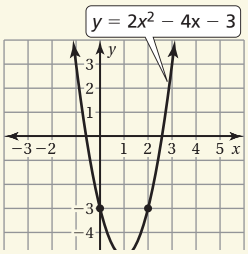
25. a. $T = \dfrac{50 - x}{(1 - 0.05x)(0.05x^2 + 5)}$

    b. 2020; This will be 20 years after 2000 and 20 is the excluded value.

26. $\dfrac{3x^2}{4(x-1)}$

Fair Game Review

27.



domain: all real numbers; range: $y \geq -5$

28–29. See Additional Answers.

30. D

Mini-Assessment

Find the product.

1. $\dfrac{3x^3}{x-8} \cdot \dfrac{8-x}{6x}$ $-\dfrac{x^2}{2}$

2. $\dfrac{2y^2 - 6y}{5y^2} \cdot \dfrac{5y^2 - 5y - 60}{y^2 - 7y + 12}$ $\dfrac{2(y+3)}{y}$

Find the quotient.

3. $\dfrac{6w}{2w-5} \div \dfrac{3}{2w-5}$ $2w$

4. $\dfrac{c^2 - 49}{3c} \div (2c - 14)$ $\dfrac{c+7}{6c}$

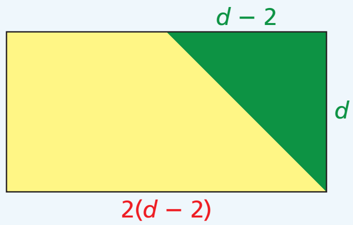
Taking Math Deeper

Exercise 23

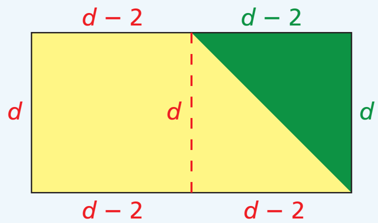
Students may realize that they can find the answer using similarity.

1. Interpret the diagram.

   Factoring the length of the campground as $2(d - 2)$ shows that the shorter side length of the shaded campsites is one-half of the length of the campground.



2. Divide the campground into two identical rectangles and label the dimensions.



3. Answer the question.

   From the diagram, you can see that each rectangle represents one-half of the area of the campground.

   You can also see that the area of the shaded campsites is one-half of the area of each rectangle, or one-quarter of the area of the campground.

   So, the probability that your campsite has shade is $\frac{1}{4}$, or 25%.

Project

Research campground regulations and fees at national parks. What regulations are shared by more than one of the parks you researched?

Reteaching and Enrichment Strategies

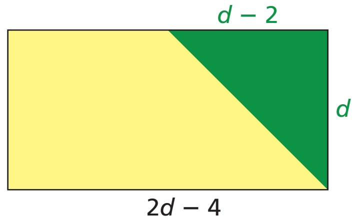
If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension • School-to-Work Start the next section

Find the product or quotient.

20. $\dfrac{2b^2 - b - 3}{b^2 - 6b - 7} \cdot \dfrac{b^2 - 3b - 28}{4b^2 - 4b - 3}$

21. $\dfrac{8y^2 + 6y - 5}{1 - 4y^2} \div \dfrac{12y^2 - y - 20}{6y^2 - 5y - 4}$

22. REASONING What are the excluded values of $\dfrac{x^2 + x - 2}{x - 3} \div \dfrac{x + 2}{x + 1}$?



23. CAMPSITE A campsite is in the shape of a rectangle. The green region represents campsites with shade. The yellow represents campsites without shade. Your campsite is randomly assigned. What is the probability that your campsite has shade?

24. TECHNOLOGY You can use a graphing calculator to check your answers when multiplying or dividing rational expressions. For instance, graph $y = \dfrac{5}{2x^3} \cdot \dfrac{4x^3}{x + 1}$ and $y = \dfrac{10}{x + 1}$ from Example 1a in the same viewing window.

a. What do you notice about the graphs?

b. How can you use the *table* feature to find the excluded values?

25. CHARITY The revenue R (in thousands of dollars) and the average ticket price P (in dollars) for a charity event can be modeled by $R = \dfrac{50 - x}{1 - 0.05x}$ and $P = 0.05x^2 + 5$, where x is the number of years since 2000. (*Note:* revenue = tickets sold × ticket price)

a. Write an equation that models the number T of tickets sold as a function of x.

b. In what year will this model become invalid? Explain your reasoning.

26. Structure Write $\dfrac{8x^3}{x + 1} \div \dfrac{2x - 2}{3x} \div \dfrac{16x^2}{x + 1}$ in simplest form.

Fair Game Review What you learned in previous grades & lessons

Graph the function. Describe the domain and range. *(Section 8.4)*

27. $y = 2x^2 - 4x - 3$

28. $y = \dfrac{1}{4}x^2 - 5x + 2$

29. $y = -4x^2 + 8x + 5$

30. MULTIPLE CHOICE What is the distance between (2, 3) and (6, 5)? *(Section 10.4)*

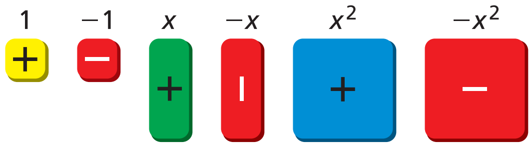
(A) $\sqrt{6}$ (B) 4 (C) $3\sqrt{2}$ (D) $2\sqrt{5}$

11.5 Dividing Polynomials

Essential Question How can you divide one polynomial by another polynomial?

1 ACTIVITY: Dividing Polynomials

Work with a partner. Six different algebra tiles are shown below.

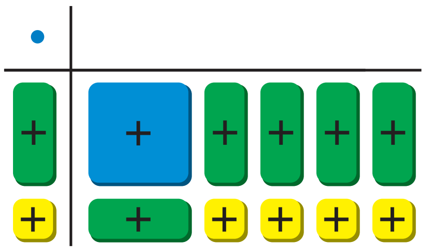


Sample:

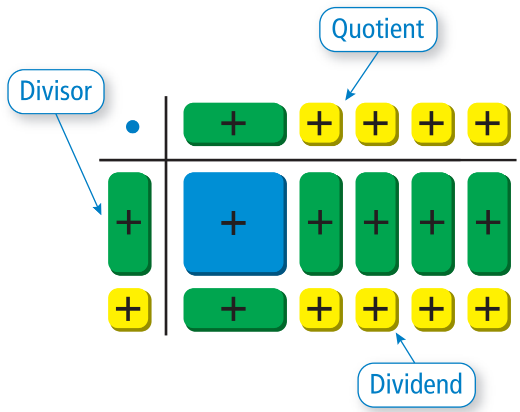
Step 1: Arrange tiles to model

$(x^2 + 5x + 4) \div (x + 1)$

in a rectangular pattern.



Step 2: Complete the pattern.



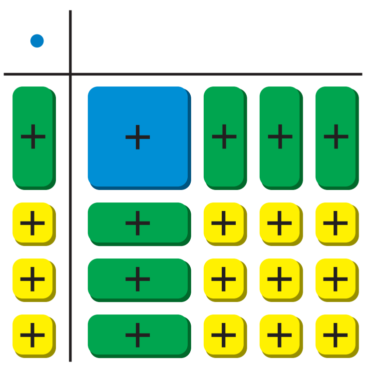
Step 3: Use the completed pattern to write

$$(x^2 + 5x + 4) \quad \div \quad (x + 1) = x + 4.$$

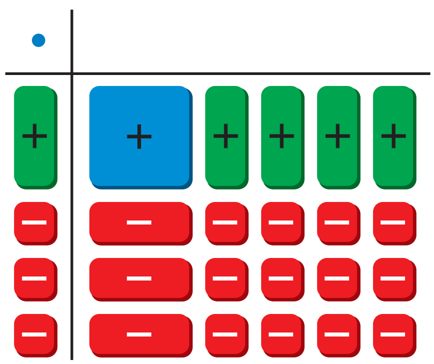
Dividend ÷ Divisor = Quotient

Complete the pattern and write the division problem.

a.



b.



Common Core

Rational Expressions

In this lesson, you will

- divide polynomials by monomials.
- divide polynomials by binomials.

Learning Standard A.SSE.2

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP7 Look for and Make Use of Structure:** Students need to understand how polynomial division and whole number division are alike. The representation and the algorithmic process are similar. Making connections to prior understanding will help students see the similarities.

Motivate

- Have students use their calculators to find the quotients.

Dividend	Divisor	Quotient
4	2	2
2**5**2	12	21
23,**6**32	112	211
2,34**7**,432	1112	2111
234,5**8**5,432	11,112	21,111
23,456,**9**65,432	111,112	211,111

? "What patterns do you observe?" The dividends are palindromes; The **middle digit** of the dividend increases by 1; Digits in the quotient are the reverse of digits in the divisor; The divisor and quotient have only 1s and 2s.

- Explain that today's activity involves dividing trinomials by binomials.

Activity Notes

Activity 1

- Work through the sample with students. From previous work, students should understand that all of the tiles must be used to form a rectangle.
- **MP2 Reason Abstractly and Quantitatively:** Arrange the tiles so that one dimension is $x + 1$. This represents the division problem. The dividend is $x^2 + 5x + 4$ and the divisor is $x + 1$. The quotient is the other dimension, $x + 4$.
- **Connection:** The divisor and quotient are factors of the dividend. This problem shows that $(x + 4)(x + 1) = x^2 + 5x + 4$.
- Students may realize that this is similar to factoring a trinomial.
- Have students try the two problems on their own. In part (b), they need to pay attention to the signs.
- Students should check their answers by multiplying.

Common Core State Standards

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Previous Learning

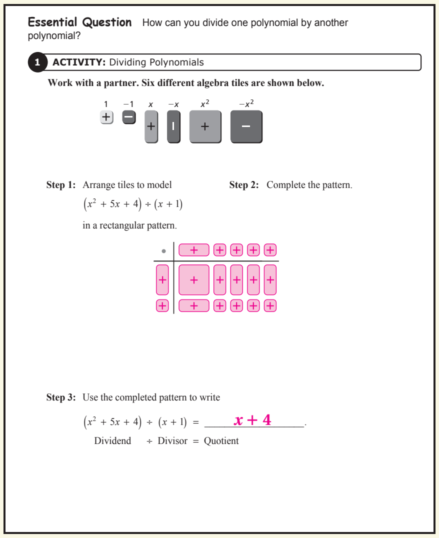
Students should know how to simplify rational expressions and use long division.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.5 Record and Practice Journal



Essential Question How can you divide one polynomial by another polynomial?

1 ACTIVITY: Dividing Polynomials

Work with a partner. Six different algebra tiles are shown below.

Step 1: Arrange tiles to model $(x^2 + 5x + 4) \div (x + 1)$ in a rectangular pattern.

Step 2: Complete the pattern.

Step 3: Use the completed pattern to write

$(x^2 + 5x + 4) \div (x + 1) =$ $x + 4$.

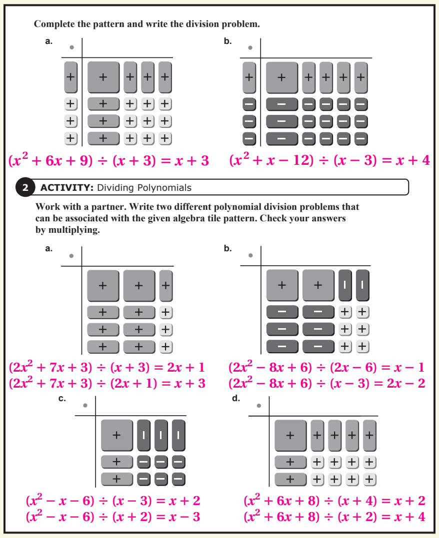
Dividend ÷ Divisor = Quotient

English Language Learners

Notebook Development

Provide English language learners a handout with the detailed solution of the example. This will allow the student to focus on the text as you work through the problem. Students should highlight on the handout as they follow along. This can be added to their notebook and referred to as they work the exercises.

11.5 Record and Practice Journal



Complete the pattern and write the division problem.

a. $(x^2 + 6x + 9) \div (x + 3) = x + 3$

b. $(x^2 + x - 12) \div (x - 3) = x + 4$

2 ACTIVITY: Dividing Polynomials

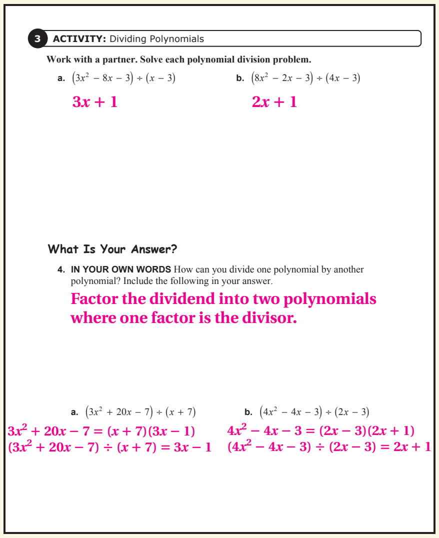
Work with a partner. Write two different polynomial division problems that can be associated with the given algebra tile pattern. Check your answers by multiplying.

a. $(2x^2 + 7x + 3) \div (x + 3) = 2x + 1$
$(2x^2 + 7x + 3) \div (2x + 1) = x + 3$

b. $(2x^2 - 8x + 6) \div (2x - 6) = x - 1$
$(2x^2 - 8x + 6) \div (x - 3) = 2x - 2$

c. $(x^2 - x - 6) \div (x - 3) = x + 2$
$(x^2 - x - 6) \div (x + 2) = x - 3$

d. $(x^2 + 6x + 8) \div (x + 4) = x + 2$
$(x^2 + 6x + 8) \div (x + 2) = x + 4$



3 ACTIVITY: Dividing Polynomials

Work with a partner. Solve each polynomial division problem.

a. $(3x^2 - 8x - 3) \div (x - 3)$
$3x + 1$

b. $(8x^2 - 2x - 3) \div (4x - 3)$
$2x + 1$

What Is Your Answer?

4. IN YOUR OWN WORDS How can you divide one polynomial by another polynomial? Include the following in your answer.

Factor the dividend into two polynomials where one factor is the divisor.

a. $(3x^2 + 20x - 7) \div (x + 7)$
$3x^2 + 20x - 7 = (x + 7)(3x - 1)$
$(3x^2 + 20x - 7) \div (x + 7) = 3x - 1$

b. $(4x^2 - 4x - 3) \div (2x - 3)$
$4x^2 - 4x - 3 = (2x - 3)(2x + 1)$
$(4x^2 - 4x - 3) \div (2x - 3) = 2x + 1$

Laurie's Notes

Activity 2

- Remind students that $a \div b = c$ can also be written as $a \div c = b$. In each case, $bc = a$.
- Make sure students name the dividend correctly in each part. In part (c), the linear terms can be simplified so the dividend is $x^2 - x - 6$.
- Tell students to check their work by multiplying the quotient and the divisor to make sure the result is the dividend.

Activity 3

- In this activity, students may decide not to model the problem with algebra tiles. They may feel comfortable enough at this point to think about what the second factor (quotient) needs to be.
- Students who found factoring to be somewhat easy should not have difficulty with these problems.

? "What clues or strategies did you use in finding the quotient?" Listen for students talking about the leading coefficients and the constant terms.

What Is Your Answer?

- **Neighbor Check:** Have students work independently and then have their neighbors check their work. Have students discuss any discrepancies.

Closure

- **Exit Ticket:** What is the quotient when you divide $2x^2 + 5x + 3$ by $x + 1$?
$2x + 3$

2 ACTIVITY: Dividing Polynomials

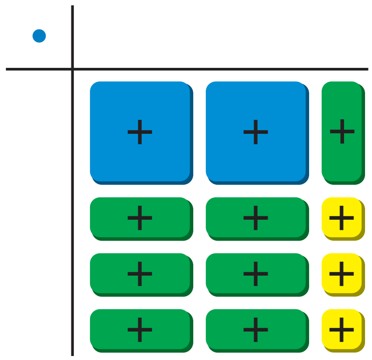
Math Practice 2

Make Sense of Quantities

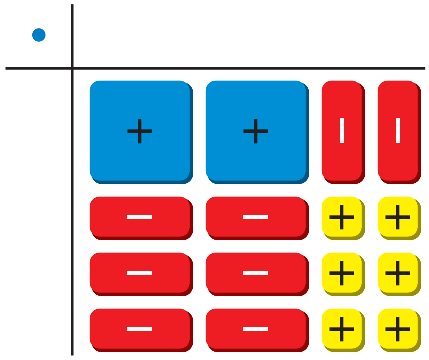
What do the algebra tiles represent? How can you use the tiles to divide one polynomial by another polynomial?

Work with a partner. Write two different polynomial division problems that can be associated with the given algebra tile pattern. Check your answers by multiplying.

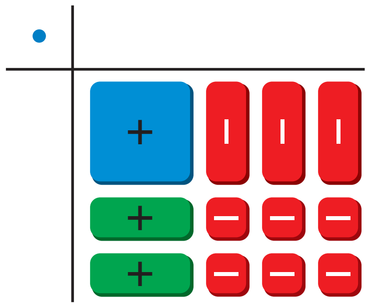
a.



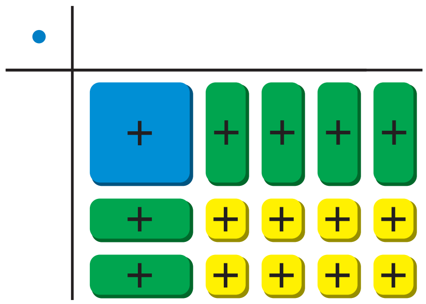
b.



c.



d.



3 ACTIVITY: Dividing Polynomials

Work with a partner. Solve each polynomial division problem.

a. $(3x^2 - 8x - 3) \div (x - 3)$

b. $(8x^2 - 2x - 3) \div (4x - 3)$

What Is Your Answer?

4. **IN YOUR OWN WORDS** How can you divide one polynomial by another polynomial? Include the following in your answer.

a. $(3x^2 + 20x - 7) \div (x + 7)$

b. $(4x^2 - 4x - 3) \div (2x - 3)$

Practice Use what you learned about dividing polynomials to complete Exercises 4 and 5 on page 578.

11.5 Lesson

Check It Out
Lesson Tutorials
BigIdeasMath.com

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 1 Dividing a Polynomial by a Monomial

Find $(3x^2 + x - 6) \div 3x$.

$$(3x^2 + x - 6) \div 3x = \frac{3x^2 + x - 6}{3x}$$ Write as a fraction.

$$= \frac{3x^2}{3x} + \frac{x}{3x} - \frac{6}{3x}$$ Divide each term by $3x$.

$$= \frac{\cancel{3}x^{\cancel{2}}}{\cancel{3}\cancel{x}} + \frac{\cancel{x}}{3\cancel{x}} - \frac{\overset{2}{\cancel{6}}}{\cancel{3}x}$$ Divide out the common factors.

$$= x + \frac{1}{3} - \frac{2}{x}$$ Simplify.

On Your Own

Now You're Ready
Exercises 6 and 7

Find the quotient.

1. $(4z^2 - 18z) \div 2z$ **2.** $(n^2 - 4n + 8) \div n$ **3.** $(y^3 - 4y^2 + 9y) \div 4y$

You can use long division to divide a polynomial by a binomial.

EXAMPLE 2 Dividing a Polynomial by a Binomial: No Remainder

Find $(m^2 + 4m + 3) \div (m + 1)$.

Step 1: Divide the first term of the dividend by the first term of the divisor.

Align like terms in the quotient and dividend.

$$\begin{array}{r} m \\ m+1 \overline{\smash{)}\, m^2 + 4m + 3} \\ \underline{m^2 + m\phantom{{}+3}} \\ 3m + 3 \end{array}$$

Divide: $m^2 \div m = m$.

Multiply: $m(m + 1)$.

Subtract. Bring down the 3.

Step 2: Divide the first term of $3m + 3$ by the first term of the divisor.

$$\begin{array}{r} m + 3 \\ m+1 \overline{\smash{)}\, m^2 + 4m + 3} \\ \underline{m^2 + m\phantom{{}+3}} \\ 3m + 3 \\ \underline{3m + 3} \\ 0 \end{array}$$

Divide: $3m \div m = 3$.

Multiply: $3(m + 1)$.

Subtract.

So, $(m^2 + 4m + 3) \div (m + 1) = m + 3$.

Study Tip

There is no remainder in Example 2, so you could have factored the dividend and divided out a common factor.

$$\frac{m^2 + 4m + 3}{m + 1} = \frac{(m + 3)\cancel{(m + 1)}}{\cancel{m + 1}} = m + 3$$

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students used algebra tiles to model polynomial division. (MP2, MP7)
- **Today:** Students will divide polynomials.

Motivate

- Ask students to describe different ways to write "48 divided by 6."
- Students should be familiar with the notations $48 \div 6$, $\frac{48}{6}$ and $6\overline{)48}$.
- In today's lesson, students will divide polynomials using these notations.

Lesson Notes

Example 1

- Write the problem as shown and then represent it as a fraction.
- ? "How do you divide a trinomial by a monomial?" Listen for students describing that you divide each term in the numerator by the denominator.
- **Big Idea:** Students should recognize how this is similar to adding and subtracting fractions with like denominators. In this problem, you start with a single fraction and then break it into three parts.
- Work through the rest of the example as shown.
- **MP7 Look for and Make Use of Structure:** Point out to students that the quotient is not a polynomial because of the $-\frac{2}{x}$ term.

On Your Own

- In Question 3, point out that $\frac{y^2}{4} = \frac{1}{4}y^2$. Ask students to explain why.

Example 2

- Say, "Now we are going to divide polynomials using long division."
- Set up the problem as shown.
- ? "How many times does m divide into m^2?" m times
- Record the m in the quotient above the m-term in the dividend. Multiply each term of the divisor by m.
- **Teaching Tip:** Although the subtraction symbol is not shown, remind students that each term of $m^2 + m$ is being subtracted.
- Continue to work the problem as shown.
- The remainder is 0, meaning that the binomial $m + 1$ divides into the trinomial evenly. Refer to the Study Tip to make this connection.

Goal Today's lesson is dividing polynomials.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Find $(4x^2 + 6x - 8) \div 2x$. $2x + 3 - \frac{4}{x}$

On Your Own

1. $2z - 9$
2. $n - 4 + \frac{8}{n}$
3. $\frac{y^2}{4} - y + \frac{9}{4}$

Extra Example 2

Find $(c^2 - 14c + 49) \div (c - 7)$. $c - 7$

Extra Example 3

Find $(d^2 - 5d + 7) \div (d - 4)$.

$d - 1 + \frac{3}{d - 4}$

On Your Own

4. $s + 4$
5. $x + 2 - \frac{9}{x + 2}$

Extra Example 4

Find $(4y^2 - 16) \div (2y + 4)$. $2y - 4$

On Your Own

6. $z - 9 + \frac{87}{z + 9}$
7. $3y - 2$

Differentiated Instruction

Visual

When dividing polynomials using long division, students may find it difficult because they are not focused on the important information in each step. Students only need to focus on the first term of the divisor and the first term of the dividend. Have them use strips of paper to cover the other terms in the expressions.

$$\begin{array}{r} 4x \\ x\,\square\,\overline{\big)\,4x^2\,\square} \end{array}$$

Laurie's Notes

Example 3

- Not all long division problems have a remainder of 0. When there is a nonzero remainder, you add $\frac{\text{remainder}}{\text{divisor}}$ to the quotient. For example, the long division below shows that $49 \div 6 = 8\frac{1}{6}$.

$$\begin{array}{r} 8 \\ 6\,\overline{)\,49} \\ \underline{48} \\ 1 \end{array}$$

- Write the problem and ask how many times y divides into y^2. Caution students about the sign when subtracting: $-7y - (-3y) = -4y$.
- Continue to work through the problem as shown, writing the remainder over the divisor.

? **Extention:** "Are polynomials closed under division?" no

On Your Own

- **Think-Pair-Share:** Students should read each question independently and then work in pairs to answer the questions. When they have answered the questions, the pair should compare their answers with another group and discuss any discrepancies.

Example 4

- Perform the long division problem $1020 \div 50$. Discuss the importance of the 0s in keeping digits in the correct place values.
- **MP6 Attend to Precision:** When dividing polynomials with missing terms, insert terms with coefficients of 0 for the same purpose.
- Write the problem. Note that the q-term is missing, so it is inserted in the dividend with a coefficient of 0.

On Your Own

- The remainder is 0 in Question 7, so the divisor is a factor of the dividend.

Closure

- **Writing Prompt:** Polynomial division and whole number division are alike because . . . the notation is similar, remainders are handled the same, etc.

When you use long division to divide polynomials and you obtain a nonzero remainder, use the following rule.

$$\text{Dividend} \div \text{Divisor} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

EXAMPLE 3 Dividing a Polynomial by a Binomial: Remainder

Find $(2 - 7y + y^2) \div (y - 3)$.

$$\begin{array}{r l}
 & \quad\;\; y - 4 \\
y - 3 \,\big)\!\! & \overline{y^2 - 7y + 2} \\
 & \underline{y^2 - 3y} \\
 & \quad\; -4y + 2 \\
 & \quad\; \underline{-4y + 12} \\
 & \qquad\quad -10
\end{array}$$

Write the dividend in standard form.

Multiply: $y(y - 3)$.

Subtract. Bring down the 2.

Multiply: $-4(y - 3)$.

Subtract.

So, $(2 - 7y + y^2) \div (y - 3) = y - 4 - \dfrac{10}{y - 3}$.

On Your Own

Now You're Ready
Exercises 8–15

Find the quotient.

4. $(s^2 - 3s - 28) \div (s - 7)$
5. $(x^2 + 4x - 5) \div (2 + x)$

EXAMPLE 4 Inserting a Missing Term

Find $(3q^2 - 8) \div (q - 2)$.

$$\begin{array}{r l}
 & \qquad\; 3q + 6 \\
q - 2 \,\big)\!\! & \overline{3q^2 + 0q - 8} \\
 & \underline{3q^2 - 6q} \\
 & \qquad\; 6q - 8 \\
 & \qquad\; \underline{6q - 12} \\
 & \qquad\qquad\;\; 4
\end{array}$$

Include a q-term with a coefficient of 0.

Multiply: $3q(q - 2)$.

Subtract. Bring down the -8.

Multiply: $6(q - 2)$.

Subtract.

Study Tip

When dividing polynomials using long division, first write the polynomials in standard form and insert any missing terms.

So, $(3q^2 - 8) \div (q - 2) = 3q + 6 + \dfrac{4}{q - 2}$.

On Your Own

Now You're Ready
Exercises 19–22

Find the quotient.

6. $(z^2 + 6) \div (z + 9)$
7. $(9y^2 - 4) \div (3y + 2)$

11.5 Exercises

Check It Out
Help with Homework
BigIdeasMath.com

Vocabulary and Concept Check

1. **WRITING** How do you divide a polynomial by a monomial? by a binomial?

2. **REASONING** How can you check your answer when dividing polynomials?

3. **NUMBER SENSE** How do you know whether a binomial is a factor of a polynomial?

Practice and Problem Solving

Use algebra tiles to find the quotient.

4. $(2x^2 + 6x - 8) \div (x + 4)$

5. $(4x^2 - 5x - 6) \div (4x + 3)$

Find the quotient.

1 6. $(8c^2 + 6c - 7) \div 8c$

7. $(3n^3 - 4n^2 + 12) \div 6n$

2 3 8. $(m^2 - 6m - 16) \div (m + 2)$

9. $(z^2 + 10z + 21) \div (z + 3)$

10. $(5y + 8) \div (y - 4)$

11. $(3h^2 + 2h - 1) \div (1 + h)$

12. $(3 - a + 2a^2) \div (a + 5)$

13. $(2 + 8k^2 - 9k) \div (k - 1)$

14. $(6x^2 + 5 + 17x) \div (1 + 3x)$

15. $(g - 7 + 6g^2) \div (2g - 3)$

ERROR ANALYSIS **Describe and correct the error in finding the quotient.**

16.

$$\begin{array}{r} 4 \\ x + 1 \overline{) 4x + 3} \\ \underline{4x + 4} \\ -1 \end{array}$$

$$(4x + 3) \div (x + 1) = 4 - \frac{1}{4x + 3}$$

17.

$$\begin{array}{r} x - 4 \\ x - 3 \overline{) x^2 - x - 6} \\ \underline{x^2 - 3x} \\ -4x - 6 \\ \underline{-4x + 12} \\ 6 \end{array}$$

$$(x^2 - x - 6) \div (x - 3) = x - 4 + \frac{6}{x - 3}$$

18. **AMUSEMENT PARK** The cost of a field trip to an amusement park is represented by $35x + 300$, where x is the number of students going on the trip. The cost is shared equally by all the students except for three students whose parents are acting as chaperones. Find $(35x + 300) \div (x - 3)$ to find an expression for how much each student pays.

Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1–3, 6–24 even, 28, 30–33	6, 12, 18, 20, 28
Advanced	1–3, 7–21 odd, 18, 26–29, 30–33	15, 18, 21, 28, 29

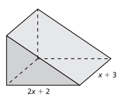
Common Errors

- **Exercises 12–15** Students may forget to write the polynomials in standard form before dividing the polynomials.
- **Exercises 19–22** Students may forget to insert missing terms before dividing the polynomials.
- **Exercise 22** Students may stop dividing too early. Remind them to add a constant of 0 to $10y^2 - 9y$ before dividing.
- **Exercise 27** Students may not multiply the length by the height before dividing because they have not seen an exercise where the divisor is a trinomial. Encourage them to use the process shown in the lesson for this new situation.
- **Exercise 28** Students may not be able to factor the polynomial because they do not recognize the polynomial as a difference of two squares.

11.5 Record and Practice Journal

Find the quotient.

1. $(b^2 + 11b + 18) \div (b + 2)$
   $b + 9$
2. $(n^2 - n - 30) \div (n + 5)$
   $n - 6$
3. $(4w^2 + 9w - 3) \div (w + 3)$
   $4w - 3 + \frac{6}{w + 3}$
4. $(5c^2 - 10 - 2c) \div (c - 1)$
   $5c + 3 - \frac{7}{c - 1}$
5. $(21 + 8x^2 - 26x) \div (4x - 7)$
   $2x - 3$
6. $(15h - 4 + 6h^2) \div (2h + 9)$
   $3h - 6 + \frac{50}{2h + 9}$
7. $(r^2 - 12) \div (r + 8)$
   $r - 8 + \frac{52}{r + 8}$
8. $(12y^2 + 8y) \div (3y - 4)$
   $4y + 8 + \frac{32}{3y - 4}$
9. The volume of the triangular prism is $x^3 + 6x^2 + 11x + 6$. Write an expression for the height of the prism.
   $x + 2$



Vocabulary and Concept Check

1. Monomial: Divide each term of the polynomial by the monomial.

   Binomial: You can use long division to divide a polynomial by a binomial.
2. The product of the quotient and divisor should be equal to the dividend.
3. When you divide the polynomial by the binomial, the remainder is 0.

Practice and Problem Solving

4. $2x - 2$
5. $x - 2$
6. $c + \frac{3}{4} - \frac{7}{8c}$
7. $\frac{n^2}{2} - \frac{2n}{3} + \frac{2}{n}$
8. $m - 8$
9. $z + 7$
10. $5 + \frac{28}{y - 4}$
11. $3h - 1$
12. $2a - 11 + \frac{58}{a + 5}$
13. $8k - 1 + \frac{1}{k - 1}$
14. $2x + 5$
15. $3g + 5 + \frac{8}{2g - 3}$
16. The remainder, -1, should be placed over the divisor, not the dividend;

    $(4x + 3) \div (x + 1) = 4 - \frac{1}{x + 1}$
17. See Additional Answers.
18. $35 + \frac{405}{x - 3}$

Practice and Problem Solving

19. $d - 3$

20. $r - 5 + \dfrac{35}{r + 5}$

21. $4n + 2 + \dfrac{5}{2n - 1}$

22. $2y - 1 - \dfrac{2}{5y - 2}$

23. The dividend is missing an x-term with a coefficient of 0 which resulted in like terms not being aligned;

$(2x^2 - 5) \div (x + 2) = 2x - 4 + \dfrac{3}{x + 2}$

24. -20

25. The sum of the degrees of the divisor and quotient are equal to the degree of the dividend.

26. **a.** The graphs coincide.

b. $x = 3, y = 3$

27. See *Taking Math Deeper.*

28. *Sample answer:* Most students will choose factoring over long division. Using long division will be messy with all the missing terms.

29. See Additional Answers.

Fair Game Review

30. $x = -1, x = 5$

31. $x = -4 + \sqrt{23}$,
$x = -4 - \sqrt{23}$

32. $x = 3 + 3\sqrt{2}, x = 3 - 3\sqrt{2}$

33. A

Mini-Assessment

Find the quotient.

1. $(12x^2 - 15x + 18) \div 3x$

$4x - 5 + \dfrac{6}{x}$

2. $(7 + 5y^2 - 2y) \div (3 + y)$

$5y - 17 + \dfrac{58}{y + 3}$

3. $(g^2 - 16) \div (g - 4)$ $g + 4$

Taking Math Deeper

Exercise 27

Students may realize that they can find the answer by observing the terms of the given expressions.

Volume of prism: $m^3 - 13m - 12$

Length of prism: $m + 3$

Height of prism: $m + 1$

1 Substitute these expressions in the formula for the volume of a rectangular prism.

$$V = \ell wh$$

$$m^3 - 13m - 12 = (m + 3)(w)(m + 1)$$

2 Determine what type of terms are in the missing expression. The expression for the volume has an m^3-term and the two known factors have an m-term. So, there must be an m-term in the missing expression.

$$m \cdot m \cdot m = m^3$$

$$m^3 - 13m - 12 = (m + 3)(m + ?)(m + 1)$$

There must be a constant term in the missing expression, otherwise there would not be a constant term in the expression for the volume.

$$m^3 - 13m - 12 = (m + 3)(m + ?)(m + 1)$$

$$3 \cdot (-4) \cdot 1 = -12$$

So, an expression for the width is $m - 4$.

3 Check your solution.

$$\begin{aligned}(m + 3)(m - 4)(m + 1) &= (m^2 - m - 12)(m + 1)\\ &= m^3 + m^2 - m^2 - m - 12m - 12\\ &= m^3 - 13m - 12\end{aligned}$$

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension • School-to-Work • Financial Literacy Start the next section

Find the quotient.

19. $(d^2 - 9) \div (d + 3)$

20. $(r^2 + 10) \div (r + 5)$

21. $(8n^2 + 3) \div (2n - 1)$

22. $(10y^2 - 9y) \div (5y - 2)$

23. **ERROR ANALYSIS** Describe and correct the error in finding the quotient.

$$
\begin{array}{r}
2x - 9 \\
x + 2 \overline{\smash{\big)}\, 2x^2 - 5} \\
\underline{2x^2 + 4x} \\
-9x \\
\underline{-9x - 18} \\
18
\end{array}
$$

$$(2x^2 - 5) \div (x + 2) = 2x - 9 + \frac{18}{x + 2}$$

24. **REASONING** Find k when $(x - 4)$ is a factor of $2x^2 - 3x + k$.

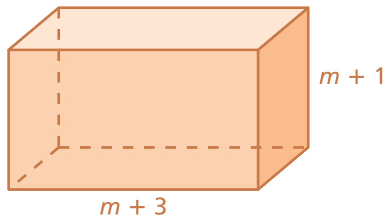
25. **CRITICAL THINKING** When dividing polynomials, how are the degrees of the dividend, divisor, and quotient related?

26. **TECHNOLOGY** Rewrite the rational function $y = \dfrac{3x - 8}{x - 3}$ in the form $y = \dfrac{a}{x - h} + k$. Graph both functions in the same viewing window of a graphing calculator.

a. What do you notice about the graphs?

b. What are the asymptotes of the graph of $y = \dfrac{3x - 8}{x - 3}$?

27. **GEOMETRY** The volume of the rectangular prism is $m^3 - 13m - 12$. Write an expression for the width of the prism.



28. **CHOOSE TOOLS** Would you use factoring or long division to simplify $\dfrac{x^8 - 1}{x - 1}$? Explain your reasoning.

29. **Repeated Reasoning** Find each quotient in the table and identify the pattern. Then predict the quotient $(x^5 - x^4 + x^3 - x^2 + x - 1) \div (x + 1)$ without calculating. Verify your prediction.

Quotient
$(x^2 - x + 1) \div (x + 1)$
$(x^3 - x^2 + x - 1) \div (x + 1)$
$(x^4 - x^3 + x^2 - x + 1) \div (x + 1)$

Fair Game Review What you learned in previous grades & lessons

Solve the equation by completing the square. *(Section 9.3)*

30. $x^2 - 4x = 5$

31. $x^2 + 8x - 7 = 0$

32. $2x^2 - 12x - 8 = 10$

33. **MULTIPLE CHOICE** What is the solution of $4^{3x} = 2^{x + 1}$? *(Section 6.4)*

Ⓐ $\frac{1}{5}$ Ⓑ $\frac{1}{2}$ Ⓒ 2 Ⓓ 3

11.6 Adding and Subtracting Rational Expressions

Essential Question How can you add and subtract rational expressions?

1 ACTIVITY: Adding Rational Expressions

Work with a partner. You and a friend have a summer job mowing lawns. Working alone it takes you 40 hours to mow all of the lawns. Working alone it takes your friend 60 hours to mow all of the lawns.

a. Write a rational expression that represents the portion of the lawns you can mow in t hours.

Portion you mow in t hours = Time · Rate

b. Write a rational expression that represents the portion of the lawns your friend can mow in t hours.

Portion your friend mows in t hours = Time · Rate

c. Add the two expressions to write a rational expression for the portion of the lawns that the two of you working together can mow in t hours.

Time · Rate + Time · Rate =

d. Use the expression in part (c) to find the total time it takes both of you working together to mow all of the lawns. Explain your reasoning.

COMMON CORE

Rational Expressions

In this lesson, you will

- add and subtract rational expressions.
- find least common denominators of two rational expressions.

Learning Standard A.SSE.2

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP1 Make Sense of Problems and Persevere in Solving Them:** The rate problems presented in this section are common applications that students often find challenging. It is important to ease into these problems so that students build understanding and make sense of the problem.

Motivate

- **Story Time:** Tell students about a job you had collecting coins out of parking meters along a roadway at the beach. Yes, it was great to work outdoors and be near the beach, however, it took 10 hours to empty all of the meters from one end of the roadway to the other.

? "What portion of the total job is done in 5 hours? in 2 hours? in t hours?" $\frac{1}{2}, \frac{1}{5}, \frac{t}{10}$

- Write on the board:

  Portion of job completed $=$ number of hours worked $\times$ rate
- In this problem, the rate is $\frac{1}{10}$.
- Because it took more than one 8-hour workday to finish the job, a second person is hired. This person is slower—it takes them 12 hours to empty all the meters.

? "How long do you think it takes both workers to empty the parking meters?" Students often take the average and incorrectly guess 11 hours. You are not looking for an answer, only a guess. Revisit this problem at the end of class.

Activity Notes

Activity 1

- This problem is similar to the one above so students should feel confident.
- **Teaching Tip:** Encourage students to use units in their answers.
- In part (c), students need to find a common denominator for 40 and 60, then simplify the fraction.

? "What does the rational expression $\frac{t}{24}$ represent?" the portion of the lawns mowed in t hours when you and your friend are working together

- Have students share their thoughts about part (d). It should seem reasonable that working together takes less time than either person working alone.

? "Explain why the answer should be between 20 and 30 hours when you and your friend are working together?" Students should refer to the two known rates. If they both could do the job in 40 hours, then together it takes 20 hours. If they both could do the job in 60 hours, then together it takes 30 hours. So, the answer should be between 20 and 30 hours.

Common Core State Standards

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Previous Learning

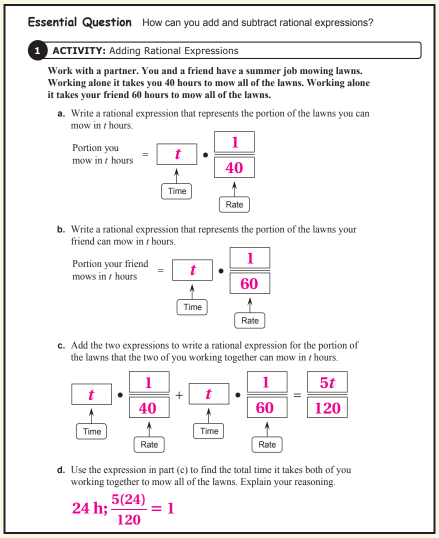
Students should know how to multiply and divide rational expressions.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.6 Record and Practice Journal



Essential Question How can you add and subtract rational expressions?

1 **ACTIVITY:** Adding Rational Expressions

Work with a partner. You and a friend have a summer job mowing lawns. Working alone it takes you 40 hours to mow all of the lawns. Working alone it takes your friend 60 hours to mow all of the lawns.

a. Write a rational expression that represents the portion of the lawns you can mow in t hours.

Portion you mow in t hours $= \underbrace{t}_{\text{Time}} \bullet \underbrace{\frac{1}{40}}_{\text{Rate}}$

b. Write a rational expression that represents the portion of the lawns your friend can mow in t hours.

Portion your friend mows in t hours $= \underbrace{t}_{\text{Time}} \bullet \underbrace{\frac{1}{60}}_{\text{Rate}}$

c. Add the two expressions to write a rational expression for the portion of the lawns that the two of you working together can mow in t hours.

$\underbrace{t}_{\text{Time}} \bullet \underbrace{\frac{1}{40}}_{\text{Rate}} + \underbrace{t}_{\text{Time}} \bullet \underbrace{\frac{1}{60}}_{\text{Rate}} = \frac{5t}{120}$

d. Use the expression in part (c) to find the total time it takes both of you working together to mow all of the lawns. Explain your reasoning.

24 h; $\frac{5(24)}{120} = 1$

Differentiated Instruction

Connection

Have students simplify the expressions $\frac{3}{4} + \frac{2}{3}$ and $\frac{7}{8} - \frac{1}{5}$ and list the steps used in the process. Make the connection between adding and subtracting rational numbers and adding and subtracting rational expressions. The process is the same in both cases. If the denominators are unlike, you must rewrite the expressions with a common denominator. Then add or subtract the numerators, and simplify.

11.6 Record and Practice Journal

2 ACTIVITY: Adding Rational Expressions

Work with a partner. You are hang gliding. For the first 10,000 feet, you travel x feet per minute. You then enter a valley in which the wind is greater, and for the next 6000 feet, you travel $2x$ feet per minute.

a. Use the formula $d = rt$ to write a rational expression that represents the time it takes you to travel the first 10,000 feet.

Time to travel first 10,000 feet $= \frac{10{,}000}{x}$ ← Distance / ← Rate

b. Use the formula $d = rt$ to write a rational expression that represents the time it takes you to travel the next 6000 feet.

Time to travel next 6000 feet $= \frac{6000}{2x}$ ← Distance / ← Rate

c. Add the two expressions to write a rational expression that represents the total time it takes you to travel 16,000 feet.

$$\frac{10{,}000}{x} + \frac{6000}{2x} = \frac{13{,}000}{x}$$

d. Use the expression in part (c) to find the total time it takes you to travel 16,000 feet when your rate during the first 10,000 feet is 2000 feet per minute.

6.5 min

What Is Your Answer?

3. **IN YOUR OWN WORDS** How can you add and subtract rational expressions? Include the following in your answer.

Rewrite (if needed) both expressions so they have a common denominator, then add or subtract the numerators over the common denominator.

a. $\frac{x}{5} + \frac{x}{10} = \frac{2x}{10} + \frac{x}{10} = \frac{3x}{10}$

b. $\frac{3}{x} + \frac{4}{x} = \frac{7}{x}$

c. $\frac{9}{x} + \frac{2}{3x} = \frac{27}{3x} + \frac{2}{3x} = \frac{29}{3x}$

d. $\frac{x}{2} - \frac{x}{4} = \frac{2x}{4} - \frac{x}{4} = \frac{x}{4}$

e. $\frac{x+1}{3} - \frac{1}{3} = \frac{x}{3}$

f. $\frac{1}{x} - \frac{1}{x^2} = \frac{x}{x^2} - \frac{1}{x^2} = \frac{x-1}{x^2}$

Laurie's Notes

Activity 2

- Before beginning this activity, review the different forms of the distance formula: $d = rt$, $r = \frac{d}{t}$, and $t = \frac{d}{r}$.
- In parts (a) and (b), students are dividing feet by $\frac{\text{feet}}{\text{minute}}$, so the answer is in minutes.
- Students should have a sense about the reasonableness of their answer. Students could find the answer in two parts: For instance, if you travel at a rate of 2000 feet per minute it takes 5 minutes to travel 10,000 feet. If your rate increases to 4000 feet per minute, it takes 1.5 minutes to travel 6000 feet. So, the time it takes to travel the entire 16,000 feet is 6.5 minutes.

What Is Your Answer?

- These six questions are practice with adding and subtracting rational expressions. Remind students that a common denominator may be needed, just as with adding and subtracting fractions.

Closure

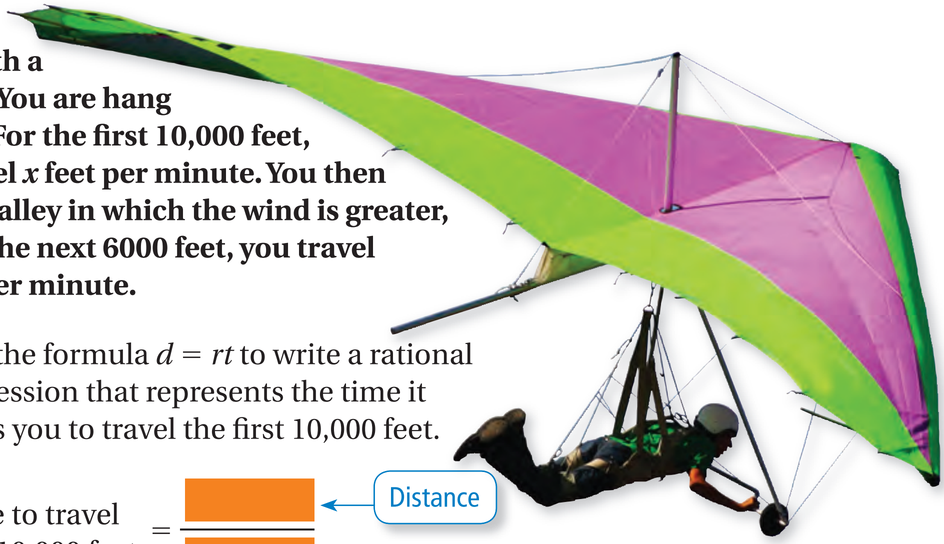
- **Exit Ticket:** Refer back to the opening problem.
  a. Write a rational expression that represents the portion of the parking meters you emptied in t hours. $\frac{t}{10}$
  b. Write a rational expression that represents the portion of the parking meters the other person emptied in t hours. $\frac{t}{12}$
  c. Write a rational expression that represents the portion of the parking meters emptied in t hours when you both work together. $\frac{11t}{60}$

2 ACTIVITY: Adding Rational Expressions

Math Practice 4

Apply Mathematics

How do the units of measure in a problem help you choose a formula? How does the formula help you write an expression?



Work with a partner. You are hang gliding. For the first 10,000 feet, you travel *x* feet per minute. You then enter a valley in which the wind is greater, and for the next 6000 feet, you travel 2*x* feet per minute.

a. Use the formula $d = rt$ to write a rational expression that represents the time it takes you to travel the first 10,000 feet.

$$\text{Time to travel first 10,000 feet} = \frac{\square}{\square}$$

Distance

Rate

b. Use the formula $d = rt$ to write a rational expression that represents the time it takes you to travel the next 6000 feet.

$$\text{Time to travel next 6000 feet} = \frac{\square}{\square}$$

Distance

Rate

c. Add the two expressions to write a rational expression that represents the total time it takes you to travel 16,000 feet.

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{\square}{\square}$$

d. Use the expression in part (c) to find the total time it takes you to travel 16,000 feet when your rate during the first 10,000 feet is 2000 feet per minute.

What Is Your Answer?

3. **IN YOUR OWN WORDS** How can you add and subtract rational expressions? Include the following in your answer.

a. $\frac{x}{5} + \frac{x}{10}$ **b.** $\frac{3}{x} + \frac{4}{x}$ **c.** $\frac{9}{x} + \frac{2}{3x}$

d. $\frac{x}{2} - \frac{x}{4}$ **e.** $\frac{x+1}{3} - \frac{1}{3}$ **f.** $\frac{1}{x} - \frac{1}{x^2}$

Practice Use what you learned about adding and subtracting rational expressions to complete Exercises 3–5 on page 585.

11.6 Lesson

Check It Out
Lesson Tutorials
BigIdeasMath.com

Key Vocabulary

least common denominator of rational expressions, *p. 583*

You can use the same rules that you used for adding and subtracting fractions to add and subtract rational expressions.

Key Idea

Adding and Subtracting Rational Expressions with Like Denominators

Let a, b, and c be polynomials, where $c \neq 0$.

Adding: $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$ Subtracting: $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$

EXAMPLE 1 Adding and Subtracting with Like Denominators

Find the sum or difference.

a. $\dfrac{5}{2x} + \dfrac{7}{2x} = \dfrac{5+7}{2x}$ Add the numerators.

$= \dfrac{12}{2x}$ Simplify.

$= \dfrac{\cancel{12}^{\,6}}{\cancel{2}x}$ Divide out the common factor.

$= \dfrac{6}{x}$ Simplify.

b. $\dfrac{3y}{y+4} - \dfrac{y-8}{y+4} = \dfrac{3y-(y-8)}{y+4}$ Subtract the numerators.

$= \dfrac{3y-y+8}{y+4}$ Use the Distributive Property.

$= \dfrac{2y+8}{y+4}$ Combine like terms.

$= \dfrac{2\cancel{(y+4)}}{\cancel{y+4}}$ Factor. Divide out the common factor.

$= 2$ Simplify.

Common Error

When subtracting rational expressions, remember to distribute the negative to each term of the numerator of the expression being subtracted.

On Your Own

Now You're Ready
Exercises 6–11

Find the sum or difference.

1. $\dfrac{4}{9z} - \dfrac{8}{9z}$ **2.** $\dfrac{3w+1}{w-1} + \dfrac{w}{w-1}$ **3.** $\dfrac{x+3}{x^2+x-2} - \dfrac{1}{x^2+x-2}$

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students wrote rational expressions and explored how to add and subtract them. (MP1)
- **Today:** Students will add and subtract rational expressions with like and unlike denominators.

Motivate

? "Has anyone ever kayaked or canoed before?" Answers will vary.

- The longest kayak race is the Yukon 1000, which is a 1000-mile race down the Yukon River in Canada and Alaska. It takes 7 to 12 days of 18 hours of paddling per day to finish the race. For comparison, it is about 1000 miles from New York City to Orlando, Florida.

Lesson Notes

Key Idea

- Write the Key Idea which states that adding and subtracting rational expressions is similar to adding and subtracting fractions. You need to find common denominators, and you should simplify your results.

Example 1

? "Do the fractions in part (a) have a common denominator?" yes "What is the sum of the numerators?" 12 "Can the rational expression be simplified? Explain." yes; The numerator and denominator have a common factor of 2.

- Write part (b). Make a point of the common error shown. Using parentheses when writing the difference of the two numerators will help students remember to distribute the negative to each term of the numerator being subtracted.

? "Can $2y + 8$ be rewritten?" yes; You can factor out a 2 to get $2(y + 4)$.

- Simplify the rational expression.

On Your Own

- Each question already has a common denominator.
- **Common Error:** In Question 2, the sum is $\frac{4w+1}{w-1}$. Students may try to divide out the w even though it is not a factor in the numerator or the denominator.
- Check to see that students have simplified their answers to Question 3.

Goal Today's lesson is adding and subtracting rational expressions.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Find the sum or difference.

a. $\frac{6}{7h} + \frac{8}{7h}$ $\frac{2}{h}$

b. $\frac{3k+2}{k-3} - \frac{2k+5}{k-3}$ 1

On Your Own

1. $-\frac{4}{9z}$
2. $\frac{4w+1}{w-1}$
3. $\frac{1}{x-1}$

Extra Example 2

Find the LCD of $\frac{x}{x^2 - 25}$ and $\frac{x^2}{2x - 10}$.

$2(x - 5)(x + 5)$

On Your Own

4. $28g^3$

5. $n(n + 1)$

6. $(t - 2)(t + 2)$

7. $x(x - 3)(x + 2)$

Extra Example 3

Find the sum $\frac{5}{3b^3} + \frac{b - 4}{12b^2}$.

$\frac{b^2 - 4b + 20}{12b^3}$

English Language Learners

Group Activity

Organize students in groups of 2 to 4 consisting of English language learners and English speakers. Provide each group with an overhead transparency and assign them a problem from the text. Students are to solve the problem and present the solution to the class. Encourage students to include visual clues in their presentation.

Laurie's Notes

Discuss

- "How do you find a common denominator for fractions with unlike denominators?" Students should be able to describe the process for finding the least common denominator.
- It is likely that students will mention finding a multiple common to both numbers. The method of using the prime factorization is less likely to be described.
- Say, "When adding or subtracting rational expressions with unlike denominators, you must first find the least common denominator (LCD). When variables are involved, the prime factorization method can be used."

Example 2

- Write the two rational expressions and say, "We need to find an expression that $10g^2$ and $12g$ both divide into."
- Write the prime factorization of each denominator. Say, "Use the greatest power of each factor that appears in either denominator to find the least common multiple (LCM) of the denominators."
- The LCM is $60g^2$.
- **Extension:** "What would you multiply $10g^2$ by to get $60g^2$?" 6 "What would you multiply $12g$ by to get $60g^2$?" $5g$

On Your Own

- Questions 5–7 are different from Question 4. Question 4 involves monomials in the denominator. Questions 5–7 involve binomials and trinomials in the denominator.
- You may need to help students think about the process of prime factorization for binomials and trinomials. They need to be factored so that the denominators can be represented as products.

Example 3

- Write the problem.
- "What is the LCM of 8 and 6?" 24 "What is the LCM of x and x^2?" x^2 "What is the LCM of $8x$ and $6x^2$?" $24x^2$
- "Each rational expression must now be rewritten using a denominator of $24x^2$. How do you rewrite equivalent fractions?" Multiply the numerator and the denominator by the same value.
- "What do you multiply $8x$ by to get $24x^2$?" $3x$ "What do you multiply $6x^2$ by to get $24x^2$?" 4
- **MP1 Make Sense of Problems and Persevere in Solving Them:** Asking each of these questions helps students make sense of the problem. Each rational expression is multiplied by 1, but how 1 is represented is different for each rational expression.
- Use colors to show how each rational expression is rewritten to have the same denominator of $24x^2$.
- Point out that $\frac{7x - 8}{24x^2}$ can be divided as shown in Section 11.5.

To add or subtract rational expressions with unlike denominators, rewrite the expressions so they have like denominators. You can do this by finding the least common multiple of the denominators, called the **least common denominator (LCD)**.

EXAMPLE 2 Finding the LCD of Two Rational Expressions

Find the LCD of $\frac{3}{10g^2}$ and $\frac{5}{12g}$.

First write the prime factorization of each denominator.

$$10g^2 = 2 \cdot 5 \cdot g^2 \qquad\qquad 12g = 2^2 \cdot 3 \cdot g$$

Use the greatest power of each factor that appears in either denominator to find the LCM of the denominators.

$$\text{LCM} = 2^2 \cdot 3 \cdot 5 \cdot g^2 = 60g^2$$

So, the LCD of $\frac{3}{10g^2}$ and $\frac{5}{12g}$ is $60g^2$.

On Your Own

Now You're Ready
Exercises 13–18

Find the LCD of the rational expressions.

4. $\frac{2}{7g}, -\frac{15}{4g^3}$

5. $\frac{8}{n}, \frac{n}{n+1}$

6. $\frac{t}{t^2-4}, \frac{9}{t-2}$

7. $\frac{x+1}{x^2-x-6}, \frac{5}{x(x-3)}$

EXAMPLE 3 Adding with Unlike Denominators

Find the sum $\frac{1}{8x} + \frac{x-2}{6x^2}$.

Because the expressions have unlike denominators, find the LCD.

$$8x = 2^3 \cdot x \qquad\qquad 6x^2 = 2 \cdot 3 \cdot x^2$$

The LCD is $2^3 \cdot 3 \cdot x^2 = 24x^2$.

$$\frac{1}{8x} + \frac{x-2}{6x^2} = \frac{1(3x)}{8x(3x)} + \frac{(x-2)(4)}{6x^2(4)} \qquad \text{Rewrite using the LCD, } 24x^2.$$

$$= \frac{3x}{24x^2} + \frac{4x-8}{24x^2} \qquad \text{Simplify.}$$

$$= \frac{3x+4x-8}{24x^2} \qquad \text{Add the numerators.}$$

$$= \frac{7x-8}{24x^2} \qquad \text{Simplify.}$$

Study Tip

To rewrite each expression using the LCD, multiply the numerator and denominator of each expression by the factor that makes its denominator the LCD.

EXAMPLE 4 Subtracting with Unlike Denominators

Find the difference $\dfrac{x+3}{x^2-8x+12} - \dfrac{2}{x-6}$.

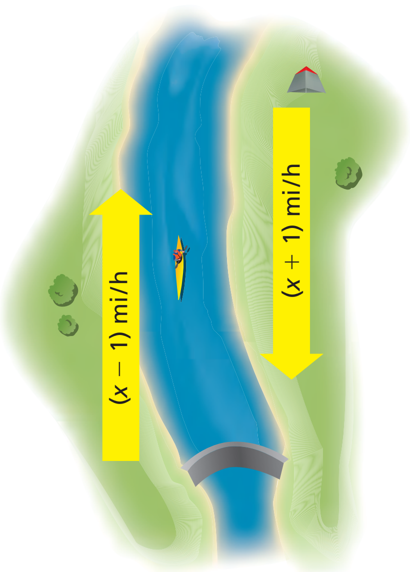
$$\frac{x+3}{x^2-8x+12} - \frac{2}{x-6} = \frac{x+3}{(x-6)(x-2)} - \frac{2}{x-6}$$ Factor $x^2 - 8x + 12$.

$$= \frac{x+3}{(x-6)(x-2)} - \frac{2(x-2)}{(x-6)(x-2)}$$ Rewrite using the LCD, $(x-6)(x-2)$.

$$= \frac{(x+3) - 2(x-2)}{(x-6)(x-2)}$$ Subtract the numerators.

$$= \frac{-x+7}{(x-6)(x-2)}$$ Simplify.

EXAMPLE 5 Real-Life Application



You row your kayak 5 miles downstream from your campsite to a dam, and then you row back to your campsite. You row x miles per hour during the entire trip, and the river current is 1 mile per hour. Write an expression for the total time of the trip.

Solving the formula $d = rt$ for time t gives $t = \dfrac{d}{r}$. Use this to write an expression for the total time of the trip.

$$\underbrace{\text{Distance downstream} \div \text{Speed downstream}}_{\text{Time downstream}} + \underbrace{\text{Distance upstream} \div \text{Speed upstream}}_{\text{Time upstream}}$$

$$\frac{5}{x+1} + \frac{5}{x-1} = \frac{5(x-1)}{(x+1)(x-1)} + \frac{5(x+1)}{(x-1)(x+1)}$$ Write an expression. Rewrite using the LCD.

$$= \frac{5(x-1) + 5(x+1)}{(x+1)(x-1)}$$ Add the numerators.

$$= \frac{10x}{(x+1)(x-1)}$$ Simplify.

On Your Own

Now You're Ready
Exercises 20–25

Find the sum or difference.

8. $\dfrac{x+5}{3x^3} - \dfrac{2}{9x^2}$

9. $\dfrac{2k}{k+1} + \dfrac{k}{k-2}$

10. $\dfrac{3y-1}{y^2-64} - \dfrac{3}{y+8}$

11. **WHAT IF?** In Example 5, the river current is 2 miles per hour. Write an expression for the total time of the trip.

Laurie's Notes

Example 4

- ? "Is $x^2 - 8x + 12$ factorable?" yes; $(x - 6)(x - 2)$
- Explain that the denominator of the second rational expression has one binomial factor in common with the denominator of the first rational expression.
- ? "What must the second rational expression be multiplied by to have the same denominator as the first rational expression?" $\frac{x-2}{x-2}$
- Remind students to distribute the negative to each term of the numerator being subtracted.
- The final answer can also be written so that the leading coefficient of the numerator is positive. Another form of the answer is $-\frac{x-7}{(x-6)(x-2)}$.

Example 5

- Ask for a volunteer to read the problem. Discuss the context—paddling downstream means you are paddling with the current and paddling upstream means you are paddling against the current. At a constant rate, the rate of the current is added going downstream and subtracted going upstream.
- ? "What are you trying to find an expression for?" the total time for the trip "Do you know the time for either direction?" no "In general, how do you solve for time?" Divide the distance by the rate.
- The distance is the same in each direction, but the rate changes. The rate downstream is $(x + 1)$. The rate upstream is $(x - 1)$.
- ? "What is the LCD for the two rational expressions?" $(x + 1)(x - 1)$
- Work through the problem as shown. When finished, evaluate the expression for a specific value of x, such as 3. You travel downstream at a rate of 4 miles per hour and upstream at a rate of 2 miles per hour. When $x = 3$, the total time is $\frac{30}{8}$, or 3.75 hours.

On Your Own

- If time is short, have students do Question 9.

Closure

- **Exit Ticket:** Find the difference $\frac{2}{t-4} - \frac{6}{t+4}$. $\frac{-4(t+8)}{(t+4)(t-4)}$

Extra Example 4

Find the difference $\frac{1}{x+4} - \frac{3-x}{x^2+2x-8}$.

$\frac{2x-5}{(x+4)(x-2)}$

Extra Example 5

You paddle a canoe 4 miles upstream and then back downstream to your starting point. You paddle downstream 50% faster than upstream due to the current. Write an expression for the total time of the trip.

$\frac{4}{r} + \frac{4}{1.5r} = \frac{10}{1.5r} = \frac{20}{3r}$

On Your Own

8. $\frac{x+15}{9x^3}$
9. $\frac{3k(k-1)}{(k+1)(k-2)}$
10. $\frac{23}{(y+8)(y-8)}$
11. $\frac{10x}{(x+2)(x-2)}$

Vocabulary and Concept Check

1. Find the LCM of the denominators in both cases.
2. Factor $x^2 - 16$ as $(x + 4)(x - 4)$. Multiply the numerator and denominator of $\frac{1}{x+4}$ by $(x - 4)$.

Practice and Problem Solving

3. $\frac{5s}{9}$
4. $\frac{r}{16}$
5. $\frac{7}{w}$
6. $\frac{8}{3y}$
7. $\frac{3}{x+2}$
8. $\frac{2z}{z-1}$
9. $\frac{3t-4}{t-1}$
10. $\frac{1}{n-2}$
11. $\frac{1}{p-1}$
12. The denominators were added.
$\frac{1}{x-3} + \frac{4}{x-3} = \frac{5}{x-3}$
13. $2x$
14. $36y$
15. $(m + 5)(m - 4)$
16. g
17. $(h + 3)(h - 1)$
18. $(s + 2)(s - 4)$
19. $\frac{S - 2\ell w}{2(\ell + w)}$

Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1, 2, 3–27 odd, 26, 28, 40–43	7, 17, 19, 23, 26, 28
Advanced	1, 2, 19–39 odd, 26, 28, 38, 40–43	19, 23, 26, 29, 37, 39

Common Errors

- **Exercises 3–11** Students may add the denominators as well as the numerators. Remind them that only the numerators are added when adding rational expressions with a common denominator.
- **Exercise 9** Students may not subtract both terms in the second numerator. Remind them to distribute the negative when subtracting a binomial.
- **Exercises 13–18** Students may multiply the denominators to find the LCD. While this process will give a common denominator, it is not necessarily the LCD. Remind them to use prime factorization to find the LCD.

11.6 Record and Practice Journal

Find the sum or difference.

1. $\frac{3}{5g} + \frac{6}{5g}$

$\frac{9}{5g}$

2. $\frac{1}{2v+3} + \frac{4}{2v+3}$

$\frac{5}{2v+3}$

3. $\frac{11m}{4m-2} - \frac{3m+2}{4m-2}$

$\frac{4m-1}{2m-1}$

4. $\frac{y^2}{y^2+y-6} - \frac{9}{y^2+y-6}$

$\frac{y-3}{y-2}$

5. $\frac{3a+1}{4a} + \frac{a-2}{6a}$

$\frac{11a-1}{12a}$

6. $\frac{k^2+8}{k-5} - k$

$\frac{5k+8}{k-5}$

7. $\frac{4x^2-x}{3x-12} + \frac{x^2+4}{4-x}$

$\frac{x+3}{3}$

8. $\frac{6}{d+5} - \frac{3d-7}{d^2+2d-15}$

$\frac{3d-11}{d^2+2d-15}$

9. You drive 45 miles from home to a relative's house and 45 miles back home. Due to construction, your speed on the way back is only 60% of your speed on the way there. Let r be your speed (in miles per hour) while driving to your relative's house. Write an expression that represents the amount of time you spend driving on your trip.

$t = \frac{120}{r}$ h

11.6 Exercises

Check It Out
Help with Homework
BigIdeasMath.com

Vocabulary and Concept Check

1. **WRITING** Explain how finding the least common denominator of two rational expressions is similar to finding the least common denominator of two numeric fractions.

2. **REASONING** Describe how to rewrite the expressions $\frac{1}{x+4}$ and $\frac{1}{x^2-16}$ so that they have the same denominator.

Practice and Problem Solving

Find the sum or difference.

3. $\frac{4s}{9} + \frac{s}{9}$

4. $\frac{r}{8} - \frac{r}{16}$

5. $\frac{2}{w} + \frac{5}{w}$

1 6. $\frac{7}{3y} + \frac{1}{3y}$

7. $\frac{5}{x+2} - \frac{2}{x+2}$

8. $\frac{2z}{4(z-1)} + \frac{6z}{4(z-1)}$

9. $\frac{3t^2}{t^2-1} - \frac{t+4}{t^2-1}$

10. $\frac{2n+3}{n^2-n-2} + \frac{-n-2}{n^2-n-2}$

11. $\frac{p-2}{p^2-5p+4} - \frac{2}{p^2-5p+4}$

12. **ERROR ANALYSIS** Describe and correct the error in adding the rational expressions.

$$\frac{1}{x-3} + \frac{4}{x-3} = \frac{5}{2x-6}$$

Find the LCD of the rational expressions.

2 13. $\frac{9}{2x}, \frac{7}{x}$

14. $\frac{1}{12y}, \frac{5}{18y}$

15. $\frac{m}{m+5}, \frac{9}{m-4}$

16. $2g, \frac{1}{g}$

17. $\frac{h}{h+3}, \frac{1}{h^2+2h-3}$

18. $\frac{s-7}{s^2-2s-8}, \frac{3}{s-4}$

19. **CEREAL** The height of a cereal box is given by $\frac{S}{2(\ell+w)} - \frac{2\ell w}{2(\ell+w)}$, where S is the surface area, ℓ is the length, and w is the width. Find the difference.

Find the sum or difference.

3 4 **20.** $\frac{x+1}{2x} + \frac{2x-1}{5x}$

21. $\frac{y-3}{6y} + \frac{y+4}{8y}$

22. $3 - \frac{c-2}{c+2}$

23. $\frac{2m}{m-7} + \frac{4}{7-m}$

24. $\frac{x+2}{x^2+3x-10} + \frac{3}{2-x}$

25. $\frac{2p+3}{p^2-7p+12} - \frac{2}{p-3}$

26. ERROR ANALYSIS Describe and correct the error in adding the rational expressions.

$$\frac{x}{x-1} + \frac{2}{x+2} = \frac{x(x-1) + 2(x+2)}{(x-1)(x+2)}$$
$$= \frac{x^2 - x + 2x + 4}{(x-1)(x+2)}$$
$$= \frac{x^2 + x + 4}{(x-1)(x+2)}$$

27. REASONING Can you find a common denominator of two rational expressions by finding the product of the denominators? Is this product always going to be the least common denominator? Justify your answers.

28. RUNNING You run 3 miles up a hill and 3 miles down the hill. You run 25% faster going down the hill than going up the hill. Let *r* be your speed (in miles per hour) while running up the hill. Write an expression that represents the amount of time you spend running on the hill.

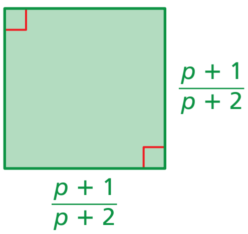
29. OPEN-ENDED Write two rational expressions with unlike denominators.

a. Find the least common denominator of the two expressions.

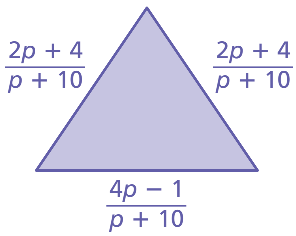
b. Add the two expressions.

Write an expression for the perimeter of the figure.

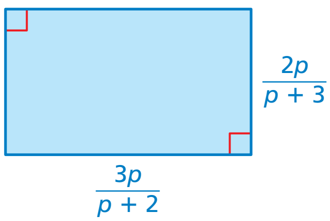
30.



31.



32.



Common Errors

- **Exercises 20–25** Students may add the denominators as well as the numerators. Remind them to find the LCD and only add the numerators when adding rational expressions.
- **Exercises 20–25** Students may multiply the numerator of each rational expression by the wrong factor of the LCD. Remind them to multiply the numerator and denominator by the factor that makes the denominator the LCD.
- **Exercises 22 and 25** Students may not subtract both terms in the second numerator. Remind them to distribute the negative when subtracting a binomial.
- **Exercises 35 and 36** Students may forget about the order of operations. Remind them of this order.

Practice and Problem Solving

20. $\dfrac{9x+3}{10x}$ **21.** $\dfrac{7}{24}$

22. $\dfrac{2c+8}{c+2}$ **23.** $\dfrac{2m-4}{m-7}$

24. $\dfrac{-2x-13}{(x+5)(x-2)}$

25. $\dfrac{11}{(p-4)(p-3)}$

26. See Additional Answers.

27. yes; not always; The product of the denominators is the product of *all* factors of both denominators. The LCM of the denominators is the product of the greatest power of each factor that appears in *either* denominator.

28. $\dfrac{5.4}{r}$

29. *Sample answer:* $\dfrac{1}{x-2}$, $\dfrac{1}{x+3}$

a. $(x-2)(x+3)$

b. $\dfrac{2x+1}{(x-2)(x+3)}$

30. $\dfrac{4p+4}{p+2}$ **31.** $\dfrac{8p+7}{p+10}$

32. $\dfrac{10p^2+26p}{(p+2)(p+3)}$

Differentiated Instruction

Organization

If students have trouble finding the least common multiple (LCM) of polynomials, have them organize the factors in a chart. For example, using the expressions $2x - 4$ and $6x^2 + 12x - 48$, line up the factors in columns.

$2 \cdot \quad (x-2)$

$\underline{2 \cdot 3 \cdot (x-2) \cdot (x+4)}$

$2 \cdot 3 \cdot (x-2) \cdot (x+4)$

The least common multiple is $6(x-2)(x+4)$.

Practice and Problem Solving

33. $\dfrac{2c+1}{c-1}$ **34.** $\dfrac{d+3}{d+5}$

35. $\dfrac{x(2x-1)}{(x+4)(x-5)}$

36. $\dfrac{2-(y+8)^2}{2(y-1)(y+8)}$

37. $\dfrac{33}{2b+20}$

38. a. $\dfrac{5x+3}{(x+3)(x-3)}$

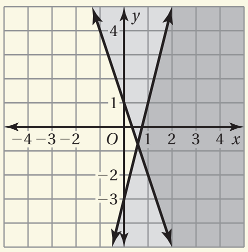
b. about 17.7 minutes

39. *Sample answer:* $\dfrac{-3}{x+4} - \dfrac{-5}{x+3}$

Fair Game Review

40–41. See Additional Answers.

42.



43. A

Mini-Assessment

1. Find the LCD of $\dfrac{x}{x-6}$ and $\dfrac{1}{x^2-3x-18}$. $(x-6)(x+3)$

Find the sum or difference.

2. $\dfrac{2n}{n^2-9} - \dfrac{n+3}{n^2-9}$ $\dfrac{1}{n+3}$

3. $\dfrac{k}{k+4} + \dfrac{6k-36}{k^2-2k-24}$ $\dfrac{k+6}{k+4}$

4. $\dfrac{w}{w-8} - \dfrac{6-3w}{w^2-10w+16}$ $\dfrac{w+3}{w-8}$

Taking Math Deeper

Exercise 39

Help students understand that they are not expected to look at this problem and immediately recognize how to solve it. Just take a deep breath, relax, and begin by looking at the given information.

1 First, rewrite the difference $\dfrac{a}{b} - \dfrac{c}{d}$: $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad}{bd} - \dfrac{bc}{bd} = \dfrac{ad-bc}{bd}$.

You can then write $\dfrac{ad-bc}{bd} = \dfrac{2x+11}{(x+4)(x+3)}$.

From this equation, $b = (x+4)$ and $d = (x+3)$. Substitute these expressions.

$$\frac{a(x+3)-(x+4)c}{(x+4)(x+3)} = \frac{2x+11}{(x+4)(x+3)}$$

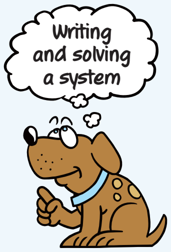
2 From this equation, you can write the following.

$a(x+3)-(x+4)c = 2x+11$	Set the numerators equal.
$ax+3a-cx-4c = 2x+11$	Distributive Property
$ax-cx+3a-4c = 2x+11$	Group like terms.
$x(a-c)+3a-4c = 2x+11$	Factor.

Write a system of equations to find a and c.

$$a - c = 2$$
$$3a - 4c = 11$$

The solution of this system is $a = -3$ and $c = -5$.



3 Check your solution.

$$\frac{-3}{(x+4)} - \frac{-5}{(x+3)} = \frac{-3(x+3)}{(x+4)(x+3)} - \frac{-5(x+4)}{(x+3)(x+4)}$$
$$= \frac{-3x-9+5x+20}{(x+4)(x+3)}$$
$$= \frac{2x+11}{(x+4)(x+3)}$$

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension • School-to-Work • Financial Literacy • Technology Connection Start the next section

Simplify the expression.

33. $\dfrac{3c+1}{c-1} + \dfrac{c+1}{c^2-4c+3} - \dfrac{c-1}{c-3}$

34. $-\dfrac{11d-8}{d^2+d-20} + \dfrac{d}{d-4} + \dfrac{5}{d+5}$

35. $\dfrac{x}{x+4} + \dfrac{x^2}{x^2-x-20} \div \dfrac{x}{x+4}$

36. $\dfrac{1}{y^2+7y-8} - \dfrac{2}{2y-2} \cdot \dfrac{y+8}{2}$

37. **HOMEWORK** You have 20 more math exercises for homework than biology exercises. You finish 15 exercises before dinner and 18 exercises after dinner. Write an expression that represents the portion of exercises that are complete.

38. **WAKEBOARDING** You are wakeboarding on a river. You travel 2 miles downstream to a marina for supplies, and then you travel 3 miles upstream to a dock. The boat travels x miles per hour during the entire trip, and the river current is 3 miles per hour.

a. Write an expression that represents the total time of the trip.

b. How long will the trip take when the speed of the boat is 18 miles per hour?

39. **Logic** Let a, b, c, and d be polynomials. Find two rational expressions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ so that $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{2x+11}{(x+4)(x+3)}$.

Fair Game Review *What you learned in previous grades & lessons*

Graph the system of linear inequalities. *(Section 4.5)*

40. $y > 2x + 1$
$y < -3x + 4$

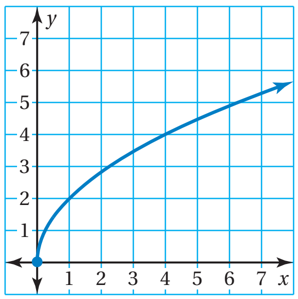
41. $y \geq -\dfrac{1}{2}x$
$y < -x - 7$

42. $y \leq 4x - 3$
$y \geq -3x + 1$

43. **MULTIPLE CHOICE** The graph of which function is shown at the right? *(Section 10.1)*

(A) $y = 2\sqrt{x}$ (B) $y = -2\sqrt{x}$

(C) $y = \dfrac{1}{2}\sqrt{x}$ (D) $y = 5\sqrt{x}$

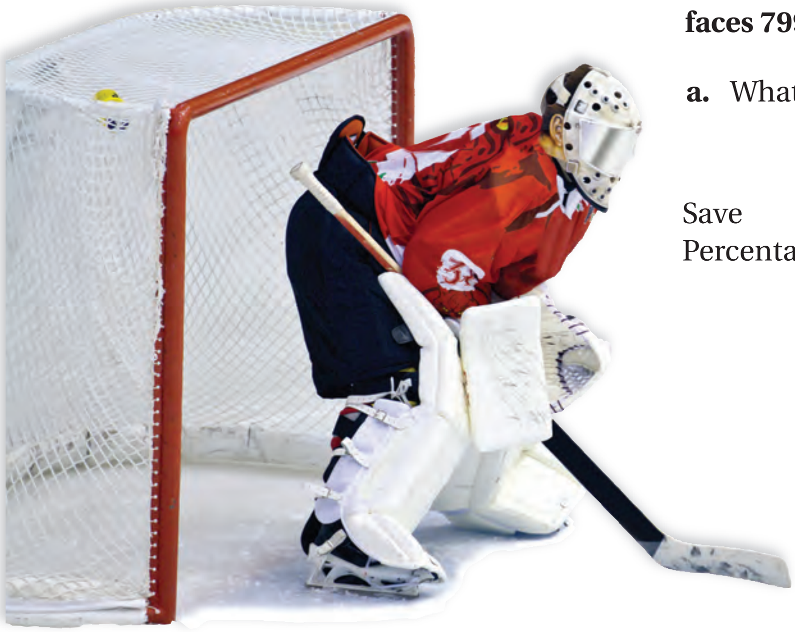


11.7 Solving Rational Equations

Essential Question How can you solve a rational equation?

1 ACTIVITY: Solving Rational Equations

Work with a partner. A hockey goalie faces 799 shots and saves 707 of them.



a. What is his save percentage?

$$\text{Save Percentage} = \frac{\square}{\square}$$

Shots saved

Shots faced

National Hockey League goalies typically have a save percentage above .900.

b. Suppose the goalie has x additional consecutive saves. Write an expression for his new save percentage.

$$\text{Save Percentage} = \frac{\square}{\square}$$

707 plus x additional saves

799 plus x additional shots faced

c. Complete the table showing the goalie's save percentage as x increases.

Additional Saves, x	0	20	40	60	80	100	120	140
Save Percentage								

d. The goalie wants to end the season with a save percentage of .900. How many additional consecutive saves must he have to achieve this? Justify your answer by solving an equation.

Common Core

Rational Functions

In this lesson, you will
- solve rational equations using cross products.
- solve rational equations using least common denominators.
- solve real-life problems.

Applying Standard A.CED.1

Laurie's Notes

Introduction

Standards for Mathematical Practice

- **MP7 Look for and Make Use of Structure:** Seeing the connection to proportions will help students understand why the Cross Products Property can be used to solve rational equations.

Motivate

- Crinkle up a piece of scrap paper into a ball.
- Let a student try 5 times to shoot the paper ball into a wastebasket from a distance of about 10 feet.
- Compute the success rate (shots in ÷ shots attempted).
- ? "Suppose [insert student name] took 5 more shots and made them all, what would happen to the success rate?" It would increase.
- Tell students that today they will investigate a similar type of problem for goalies and baseball players.

Activity Notes

Activity 1

- Ask a student to explain the role of a goalie in sports.
- Students should use a calculator to find the percentages in this activity.
- Make sure students write the correct expressions in part (b).
- ? "What do you observe about your answers in part (c)?" increasing
- **MP2 Reason Abstractly and Quantitatively:** Have a discussion about the value of $\frac{707 + x}{799 + x}$ as x increases.
  - The numerator and denominator are each increasing by the same amount each time. The ratio is increasing, but at a decreasing rate.
  - The ratio is getting closer to 1 but it will not reach 1. Connect this to the concept of an asymptote.
- **MP1a Make Sense of Problems:** Ask for volunteers to describe how they answered part (d). They may have:
  - used trial and error on a calculator.
  - set the ratio equal to 0.9, rewritten it as $\frac{9}{10}$, and then used the Cross Products Property.
  - set the ratio equal to 0.9, multiplied both sides of the equation by $(799 + x)$, and solved.
- **MP5 Use Appropriate Tools Strategically:** If time permits, use a graphing calculator to generate a table of values for $y = \frac{707 + x}{799 + x}$.

Common Core State Standards

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

Previous Learning

Students should know how to add, subtract, multiply, and divide rational expressions.

Technology for the Teacher

Dynamic Classroom

Lesson Plans
Complete Materials List

11.7 Record and Practice Journal

Essential Question How can you solve a rational equation?

1 ACTIVITY: Solving Rational Equations

Work with a partner. A hockey goalie faces 799 shots and saves 707 of them.

a. What is his save percentage?

$$\text{Save Percentage} = \frac{\boxed{707}}{\boxed{799}} \quad \begin{matrix}\leftarrow \text{Shots saved} \\ \leftarrow \text{Shots faced}\end{matrix}$$

b. Suppose the goalie has x additional consecutive saves. Write an expression for his new save percentage.

$$\text{Save Percentage} = \frac{707 + x}{799 + x} \quad \begin{matrix}\leftarrow \text{707 plus } x \text{ additional saves} \\ \leftarrow \text{799 plus } x \text{ additional shots faced}\end{matrix}$$

c. Complete the table showing the goalie's save percentage as x increases.

Additional Saves, x	0	20	40	60	80	100	120	140
Save Percentage	0.885	0.888	0.890	0.893	0.895	0.898	0.900	0.902

d. The goalie wants to end the season with a save percentage of .900. How many additional consecutive saves must he have to achieve this? Justify your answer by solving an equation.

121;

$$\frac{707 + x}{799 + x} = 0.900$$

$$x = 121$$

English Language Learners

Pair Activity

Pair English language learners with English speakers. Assign each person a problem. When they have completed their problems, they explain their solution to their partner who follows along. This will engage English language learners in conversation and help with understanding math concepts.

11.7 Record and Practice Journal

2 ACTIVITY: Solving Rational Equations

Work with a partner. A baseball player has been at bat 47 times and has 8 hits.

a. What is his batting average?

Batting Average $= \dfrac{\boxed{8}}{\boxed{47}}$

b. Suppose the player has x additional consecutive hits. Write an expression for his new batting average.

Batting Average $= \dfrac{\boxed{}\ 8 + x}{\boxed{}\ 47 + x}$

c. Complete the table showing the player's batting average as x increases.

Additional Hits, x	0	1	2	3	4	5	6	7
Batting Average	0.170	0.188	0.204	0.220	0.235	0.250	0.264	0.278

d. The player wants to end the season with a batting average of .250. How many additional consecutive hits must he have to achieve this? Justify your answer by solving an equation.

5;

$$\frac{8 + x}{47 + x} = 0.250$$

$$x = 5$$

What Is Your Answer?

3. IN YOUR OWN WORDS How can you solve a rational equation? Include the following in your answer.

Use the Cross Products Property.

a. $\dfrac{x - 6}{6} = \dfrac{2}{3}$

$3(x - 6) = 2(6)$

$x = 10$

b. $\dfrac{x + 56}{6} = \dfrac{1}{2}$

$2(x + 56) = 1(6)$

$x = -53$

c. $\dfrac{x}{4} + \dfrac{x}{2} = \dfrac{2x}{3}$

$\dfrac{3x}{4} = \dfrac{2x}{3}$

$3(3x) = 4(2x)$

$x = 0$

Laurie's Notes

Activity 2

- Discuss briefly how to find a batting average. If you are not familiar with how "at bats" are defined, perhaps one of your students will explain. For example, when a batter walks, sacrifices, or gets hit by a pitch, it is not considered an "at bat."
- This activity is very similar to the first activity in that students explore ratios where the numerator and denominator are changing by the same amount each time.
- **MP2:** Have a conversation similar to the previous activity about how the ratio is changing as x increases.
- **MP1a:** Ask for volunteers to describe how they answered part (d). They may have used methods similar to those listed in Activity 1.
- **MP5:** If time permits, use a graphing calculator to generate a table of values for $y = \dfrac{8 + x}{47 + x}$.

What Is Your Answer?

- There are several ways in which students might solve the three problems shown. Having discussed a variety of methods in class will deepen student understanding.

Closure

- Ask a question related to the Motivate. For example, if the student had a success rate of 0.8, ask how many additional consecutive shots must be made to have a scoring rate of 0.9.

2 ACTIVITY: Solving Rational Equations

Math Practice 8

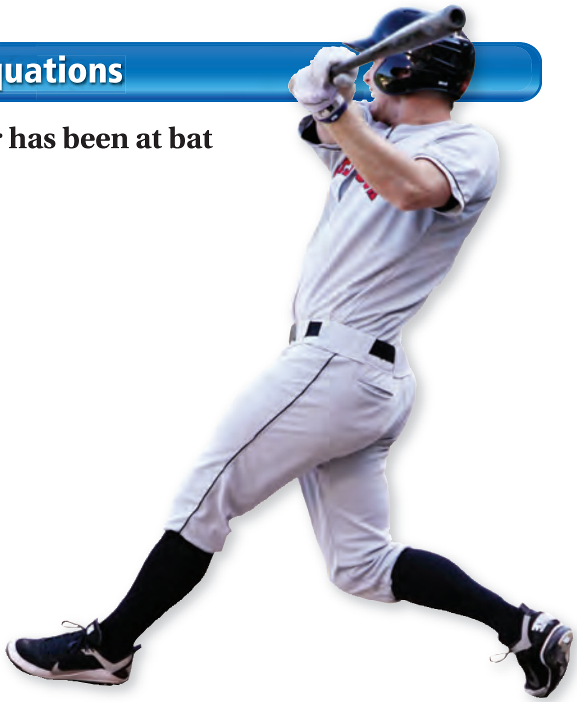
Find General Methods

What method did you use to complete the table? How can you use this information to write and solve an equation in part (d)?

Work with a partner. A baseball player has been at bat 47 times and has 8 hits.

a. What is his batting average?

Batting Average $= \dfrac{\boxed{}}{\boxed{}}$



b. Suppose the player has x additional consecutive hits. Write an expression for his new batting average.

Batting Average $= \dfrac{\boxed{}}{\boxed{}}$

The league batting average in Major League Baseball is usually between .250 and .270.

c. Complete the table showing the player's batting average as x increases.

Additional Hits, x	0	1	2	3	4	5	6	7
Batting Average								

d. The player wants to end the season with a batting average of .250. How many additional consecutive hits must he have to achieve this? Justify your answer by solving an equation.

What Is Your Answer?

3. IN YOUR OWN WORDS How can you solve a rational equation? Include the following in your answer.

a. $\dfrac{x-6}{6} = \dfrac{2}{3}$ **b.** $\dfrac{x+56}{6} = \dfrac{1}{2}$ **c.** $\dfrac{x}{4} + \dfrac{x}{2} = \dfrac{2x}{3}$

Practice Use what you learned about solving rational equations to complete Exercise 4 on page 592.

11.7 Lesson

Check It Out
Lesson Tutorials
BigIdeasMath.com

Key Vocabulary
rational equation, *p. 590*

A **rational equation** is an equation that contains rational expressions. One way to solve rational equations is to use the Cross Products Property. You can use this method when each side of a rational equation consists of one rational expression.

EXAMPLE 1 Solving Rational Equations Using Cross Products

Solve each equation.

a. $\dfrac{5}{x+4} = \dfrac{4}{x-4}$

$\dfrac{5}{x+4} = \dfrac{4}{x-4}$	Write the equation.
$5(x-4) = 4(x+4)$	Cross Products Property
$5x - 20 = 4x + 16$	Distributive Property
$5x = 4x + 36$	Add 20 to each side.
$x = 36$	Subtract $4x$ from each side.

Check

$$\frac{5}{x+4} = \frac{4}{x-4}$$

$$\frac{5}{36+4} \stackrel{?}{=} \frac{4}{36-4}$$

$$\frac{1}{8} = \frac{1}{8} \checkmark$$

b. $\dfrac{5}{y} = \dfrac{y-2}{7}$

$\dfrac{5}{y} = \dfrac{y-2}{7}$	Write the equation.
$5(7) = y(y-2)$	Cross Products Property
$35 = y^2 - 2y$	Simplify.
$0 = y^2 - 2y - 35$	Subtract 35 from each side.
$0 = (y-7)(y+5)$	Factor.
$y - 7 = 0 \quad \text{or} \quad y + 5 = 0$	Zero-Product Property
$y = 7 \quad \text{or} \quad y = -5$	Solve for y.

Check

$\dfrac{5}{7} \stackrel{?}{=} \dfrac{7-2}{7}$	Substitute for y.	$\dfrac{5}{-5} \stackrel{?}{=} \dfrac{-5-2}{7}$
$\dfrac{5}{7} = \dfrac{5}{7} \checkmark$	Simplify.	$-1 = -1 \checkmark$

On Your Own

Now You're Ready
Exercises 5–10

Solve the equation. Check your solution(s).

1. $\dfrac{2}{x-3} = \dfrac{4}{x-7}$ **2.** $\dfrac{4}{z+4} = \dfrac{z}{z+1}$ **3.** $\dfrac{3y}{4} = \dfrac{6}{y+7}$

Laurie's Notes

Introduction

Connect

- **Yesterday:** Students explored rational equations. (MP1a, MP2, MP5, MP7)
- **Today:** Students will solve rational equations.

Motivate

- Ask if any of your students have collectible cards (sports cards, card games, etc). These cards are often traded among collectors.
- Share some trivia with your students. For instance, you could discuss one of the rarest and most valuable baseball cards of all time—the Honus Wagner "T206" card. An owner of the Arizona Diamondbacks purchased the card for $2.8 million in 2007!
- Tell students they will solve a collectible card problem in this lesson.

Lesson Notes

Example 1

? "Are there any excluded values that should be noted?" yes, $x = 4$, $x = -4$

? "What property might be helpful in solving this equation?" Cross Products Property

- Remind students to include parentheses around each binomial when cross multiplying. The Distributive Property must now be used.

? "How can you check the solution?" by substitution

- **Teaching Tip**: Although the problem can be checked graphically, it is not always obvious where the graphs of two rational functions intersect. Algebraic checks are often more efficient.
- Write the equation in part (b) and ask students how it differs from the equation in part (a).

? "Are there any excluded values that should be noted?" yes, $y = 0$

- Check both solutions to be sure that neither is extraneous.

On Your Own

- **Think-Pair-Share:** Students should read each question independently and then work in pairs to answer the questions. When they have answered the questions, the pair should compare their answers with another group and discuss any discrepancies.

Goal Today's lesson is solving **rational equations**.

Technology for the Teacher

Dynamic Classroom

Lesson Tutorials
Lesson Plans
Answer Presentation Tool

Extra Example 1

Solve each equation.

a. $\frac{3}{x-2} = \frac{6}{x+1}$ $x = 5$

b. $\frac{y+5}{2} = \frac{3}{y}$ $y = -6, y = 1$

On Your Own

1. $x = -1$
2. $z = -2, z = 2$
3. $y = -8, y = 1$

Extra Example 2

Solve $\frac{x}{x-4} - \frac{3}{4} = \frac{2}{x-4}$. $x = -4$

Extra Example 3

In Example 3, you add creature cards to the deck until it contains 45% creature cards. How many do you add? 10 cards

On Your Own

4. $p = -9$
5. $n = 3$
6. $a = -7, a = 3$
7. 5 cards

Differentiated Instruction

Connection

Show students that an equation solved using the Cross Products Property can be solved by multiplying both sides by the LCD. For instance, solving Example 1(b) would look like this.

$$\frac{5}{y} = \frac{y-2}{7}$$

$$7 \cdot \cancel{y} \cdot \frac{5}{\cancel{y}} = \cancel{7} \cdot y \cdot \frac{y-2}{\cancel{7}}$$

$$7(5) = y(y-2)$$

After multiplying by the LCD and simplifying, the resulting equation is the same as if the Cross Products Property had been used. Students who have a difficult time deciding which method to use should multiply by the LCD.

Laurie's Notes

Discuss

- **MP7 Look for and Make Use of Structure:** Rational equations can involve operations on one or both sides. Multiplying each side by the LCD of the expressions eliminates the denominators. Demonstrate this by multiplying each side of $\frac{x}{2} + \frac{3}{4} = \frac{5}{8}$ by 8.

Example 2

- This example demonstrates another way to solve rational equations.
- ? "What is the LCD of the expressions in this equation?" $3(z-2)$
- ? "Are there any excluded values that should be noted?" yes, $z = 2$
- **Big Idea:** When you multiply the left side of the equation by $3(z-2)$, you use the Distributive Property. Work through this step slowly, showing how the common factors divide out.
- **Alternate Approach:** Subtract $\frac{z}{z-2}$ from each side and simplify. Then use the Cross Products Property.
- Discuss why the rational equation has no solution.

Example 3

- **MP1a Make Sense of Problems:** Ask a volunteer to read the problem. Make sure that students understand the context, especially that only creature cards are added to deck.
- This example should remind students of the problems in the activity.
- Point out to students that using the Cross Products Property and multiplying each side by the LCD results in the same equation.
- Take time to check the solution.

On Your Own

- Have volunteers share their solutions at the board.

Closure

- **Exit Ticket:** Write a rational equation that you would solve using (a) the Cross Products Property and (b) the LCD.

  Sample answers: (a) $\frac{1}{3} = \frac{x-1}{5}$ (b) $\frac{2}{x-1} + \frac{1}{3} = \frac{8}{x-1}$

When there is more than one rational expression on one or both sides of a rational equation, multiply each side by the LCD and then solve.

EXAMPLE 2 Solving a Rational Equation Using the LCD

Solve $\dfrac{z}{z-2} - \dfrac{2}{3} = \dfrac{2}{z-2}$.

$$3(z-2) \cdot \left(\frac{z}{z-2} - \frac{2}{3}\right) = 3(z-2) \cdot \frac{2}{z-2}$$ Multiply each side by the LCD, $3(z-2)$.

$$\frac{z \cdot 3\cancel{(z-2)}}{\cancel{z-2}} - \frac{2 \cdot \cancel{3}(z-2)}{\cancel{3}} = \frac{2 \cdot 3\cancel{(z-2)}}{\cancel{z-2}}$$ Multiply. Then divide out common factors.

$$3z - 2z + 4 = 6$$ Simplify.

$$z = 2$$ Solve for z.

Because each side of the equation is undefined when $z = 2$, it is an extraneous solution.

∴ The equation has no solution.

EXAMPLE 3 Real-Life Application

Your starter deck for a collectible card game has 50 cards. The deck contains 17 creature cards. You add creature cards to the deck until it contains 50% creature cards. How many do you add?

Write an equation for the ratio of creature cards to total cards after adding x creature cards.

Creature cards

Total cards

Desired percent of creature cards

$$\frac{x+17}{x+50} = 0.5$$

$$0.5(x+50) = x + 17$$ Cross Products Property

$$0.5x + 25 = x + 17$$ Distributive Property

$$8 = 0.5x$$ Simplify.

$$16 = x$$ Divide each side by 0.5.

∴ You add 16 creature cards to the deck.

On Your Own

Now You're Ready
Exercises 13–18

Solve the equation. Check your solution(s).

4. $\dfrac{1}{p} - \dfrac{2}{3} = \dfrac{7}{p}$ **5.** $\dfrac{2}{n} + \dfrac{1}{n+3} = \dfrac{5}{n+3}$ **6.** $\dfrac{4}{a-6} + 1 = \dfrac{9}{a^2-36}$

7. **WHAT IF?** In Example 3, you add creature cards until the deck contains 40% creature cards. How many do you add?

11.7 Exercises

Check It Out
Help with Homework
BigIdeasMath.com

Vocabulary and Concept Check

1. **VOCABULARY** Describe two methods for solving rational equations.

2. **OPEN-ENDED** Write a rational equation that can be solved by multiplying each side by $2x(x + 1)$.

3. **WRITING** Why should you check the solutions of a rational equation?

Practice and Problem Solving

4. A basketball player attempts 64 free throws and makes 50 of them.
   a. What is her free throw percentage?
   b. Suppose the player makes x additional consecutive free throws. Write an expression for her new free throw percentage.
   c. The player wants to end the season with a free throw percentage of .800. How many additional consecutive free throws must she make to achieve this?

Solve the equation. Check your solution(s).

1 5. $\dfrac{2}{b} = \dfrac{6}{b+2}$ 6. $\dfrac{2}{x-1} = \dfrac{3}{x+1}$ 7. $\dfrac{4}{m-4} = \dfrac{m}{3}$

8. $\dfrac{z-1}{8} = \dfrac{z}{z+9}$ 9. $\dfrac{k}{2k+5} = \dfrac{1}{k-2}$ 10. $\dfrac{3w}{w+1} = \dfrac{w}{3-w}$

11. **ERROR ANALYSIS** Describe and correct the error in solving the equation.

$$\frac{x}{x+1} = \frac{2}{x+1}$$
$$x(x+1) = 2(x+1)$$
$$x^2 + x = 2x + 2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad \text{or} \quad x = -1$$

So, the solutions are $x = 2$ and $x = -1$.

12. **WATER RESCUE** The table shows information about a water rescue team.
    a. Solve the rational equation $\dfrac{4}{x} = \dfrac{7}{x+6}$ to find the upstream speed of the rescue team.
    b. What is the downstream speed of the rescue team?

Water Rescue			
Direction	Distance	Rate	Time
Upstream	4 miles	x mi/h	t hours
Downstream	7 miles	$(x + 6)$ mi/h	t hours

Assignment Guide and Homework Check

Level	Assignment	Homework Check
Average	1–3, 5–19 odd, 12, 20, 25–28	7, 11, 12, 15, 20
Advanced	1–3, 6–18 even, 19–24, 25–28	12, 16, 20, 22, 24

Common Errors

- **Exercises 5–10** Students may multiply the numerators and multiply the denominators of the rational expressions instead of finding the cross products. Remind them of the Cross Products Property.
- **Exercises 13–18** Students may not multiply each side of the equation by the LCD. Remind them of the Multiplication Property of Equality.
- **Exercises 22–24** Students may have difficulty setting up the equations for these exercises. Remind them that the sum of the portions for everyone working is the portion of the job completed in one unit of time.

11.7 Record and Practice Journal

Solve the equation. Check your solution.

1. $\frac{4}{h-3} = \frac{8}{h}$
   $h = 6$
2. $\frac{6}{q-2} = \frac{5}{q-1}$
   $q = -4$
3. $\frac{m}{m+3} = \frac{5}{m+7}$
   $m = -5, 3$
4. $\frac{c-3}{5c-6} = \frac{c}{c-3}$
   $c = \pm\frac{3}{2}$
5. $\frac{6}{z-3} - \frac{3}{z} = \frac{6}{z}$
   $z = 9$
6. $\frac{4}{k} + \frac{14}{k+5} = \frac{8}{k+5}$
   $k = -2$
7. $\frac{d}{d+3} + \frac{1}{d-1} = \frac{4}{d^2+2d-3}$
   $d = -1$
8. $\frac{t}{t-7} - \frac{5}{t-4} = \frac{3t+3}{t^2-11t+28}$
   $t = 8$
9. An academic challenge team has 24 members, 9 of which are boys. The team is required to have 50% boys and 50% girls for a competition. How many boys does the team need to add to reach this proportion?
   6 boys

Vocabulary and Concept Check

1. Use the Cross Products Property or multiply each side by the LCD.
2. *Sample answer:* $\frac{1}{x} + \frac{6}{x+1} = \frac{x}{2}$
3. The solution may be extraneous.

Practice and Problem Solving

4. **a.** 0.78125, or about 0.781
   b. $\frac{50+x}{64+x}$
   c. 6 free throws
5. $b = 1$
6. $x = 5$
7. $m = -2$, $m = 6$
8. $z = -3$, $z = 3$
9. $k = -1$, $k = 5$
10. $w = 0$, $w = 2$
11. The solutions were not checked in the original equation. The solution $x = -1$ is extraneous because it is an excluded value. The only solution is $x = 2$.
12. **a.** 8 mi/h
    b. 14 mi/h

Practice and Problem Solving

13. $c = 5$

14. $y = -1$

15. no solution

16. $n = -3, n = \frac{2}{7}$

17. $a = -9, a = 6$

18. $x = -1$

19. Rewrite the left side using a common denominator of x. Then use the Cross Products Property to solve.

20. 4 pints

21. See *Taking Math Deeper.*

22. 4.8 minutes

23. 1.2 hours

24. 12 hours

Fair Game Review

25. $x = -4, x = 8$

26. $x = 2, x = 5$

27. $x = -2\frac{1}{5}, x = \frac{3}{5}$

28. B

Mini-Assessment

Sole the equation. Check your solution.

1. $\frac{7}{a} = \frac{1}{a-6}$ $a = 7$

2. $\frac{b}{b+6} = \frac{3}{b}$ $b = -3, b = 6$

3. $\frac{2c}{c+2} - 5 = \frac{7c}{c+2}$ $c = -1$

Taking Math Deeper

Exercise 21

As with many real-life problems, this one has a lot of information. Read through the problem and organize the given information.

1

- The club pays \$540 for a bus. So, the original bus fare per person for x club members is $\frac{540}{x}$.
- Seven hikers join the trip, so the bus fare per person is $\frac{540}{x+7}$.
- After the 7 hikers join, the bus fare per person decreases by \$7. So, the bus fare per person can also be written as $\frac{540}{x} - 7$.

2 Set the expressions for the bus fare per person equal and solve for x.

$$\frac{540}{x+7} = \frac{540}{x} - 7$$
$$\frac{540}{x+7} = \frac{540 - 7x}{x}$$
$$540x = (x+7)(540 - 7x)$$
$$540x = 540x - 7x^2 + 3780 - 49x$$
$$7x^2 + 49x - 3780 = 0$$
$$7(x+27)(x-20) = 0$$
$$x + 27 = 0 \quad \text{or} \quad x - 20 = 0$$
$$x = -27 \quad \text{or} \quad x = 20$$

The negative solution does not make sense in this situation. So, 20 members of the rappelling club are going on the trip.

3 Check your solution in the original equation: $\frac{540}{20+7} = \frac{540}{20} - 7$

$$20 = 20$$

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials Skills Review Handbook	Resources by Chapter • Enrichment and Extension • School-to-Work • Financial Literacy • Technology Connection • Life Connections Start the next section

Solve the equation. Check your solution(s).

2 **13.** $\frac{4}{5} - \frac{1}{c} = \frac{3}{c}$

14. $\frac{2}{y+3} - \frac{5}{y} = \frac{12}{y+3}$

15. $\frac{10}{d(d-2)} + \frac{4}{d} = \frac{5}{d-2}$

16. $\frac{n}{n-2} + \frac{2}{5} = \frac{1}{n+4}$

17. $\frac{6}{a+5} + 2 = \frac{28}{a^2-25}$

18. $\frac{x}{x+7} + \frac{3}{x-6} = \frac{2x+27}{x^2+x-42}$

19. REASONING Explain how you can use the Cross Products Property to solve $\frac{3}{x} + 1 = \frac{8}{x-3}$.

20. PAINT A department store paint mixer contains 4 pints of equal amounts of yellow and red paint. The shade of red that you want requires a paint mixture that is 75% red and 25% yellow. How many pints of red paint need to be added to the paint mixer?

21. RAPPELLING A rappelling club charters a bus for a trip to the mountains for \$540. To lower the bus fare per person, the club invites some hikers on the trip. After 7 hikers join the trip, the bus fare per person decreases by \$7. How many members of the rappelling club are going on the trip?

To solve *work problems*, find the portion of the job each person completes in 1 unit of time. The sum of these portions is the portion of the job completed in 1 unit of time.

22. You can mop a floor in 8 minutes. Your friend can mop the same floor in 12 minutes. Working together, how much time does it take to mop the floor?

23. You can mow a lawn in 3 hours. Your friend can mow the same lawn in 2 hours. Working together, how much time does it take to mow the lawn?

24. Reasoning A roofing contractor can shingle a roof in half the time it takes his assistant. Working together, they can shingle the roof in 8 hours. How much time does it take the roofing contractor to finish the job alone?

Fair Game Review What you learned in previous grades & lessons

Solve the equation. Check your solutions. *(Section 1.3)*

25. $|x-2| = 6$

26. $3|2x-7| = 9$

27. $2|5x+4| - 1 = 13$

28. MULTIPLE CHOICE What is the solution of $-2 < -x + 5 \le 8$? *(Section 3.4)*

Ⓐ $-7 < x \le 3$ Ⓑ $7 > x \ge -3$ Ⓒ $x \le -3$ and $x > 7$ Ⓓ $x < 7$ or $x \ge -3$

11.4–11.7 Quiz

Check It Out
Progress Check
BigIdeasMath.com

Find the product or quotient. *(Section 11.4)*

1. $\dfrac{c+2}{5c^3} \cdot \dfrac{4c^4}{6}$

2. $\dfrac{4ab^3 - 2b^3}{2a^3 + 4a^2} \cdot \dfrac{2a^2 + 4a}{4ab - 2b}$

3. $\dfrac{3k}{k+3} \div \dfrac{15}{k+3}$

4. $\dfrac{m^2 - 36}{m^3} \div \dfrac{m^2 + 12m + 36}{m^3}$

Find the quotient. *(Section 11.5)*

5. $(6j^3 + 12j^2 + 18j) \div 6j$

6. $(m^2 - 14m + 49) \div (m - 7)$

7. $(d^2 - 5d + 8) \div (d - 3)$

8. $(5n^2 + 7) \div (n - 1)$

Find the sum or difference. *(Section 11.6)*

9. $\dfrac{5}{v+1} - \dfrac{10}{v+1}$

10. $\dfrac{4r}{2r-3} + \dfrac{5r-1}{3-2r}$

11. $\dfrac{t^2 - 8}{6t} + \dfrac{-t^2 + 7}{4t}$

12. $\dfrac{3p+10}{p^2+p-20} - \dfrac{2}{p-4}$

Solve the equation. Check your solution. *(Section 11.7)*

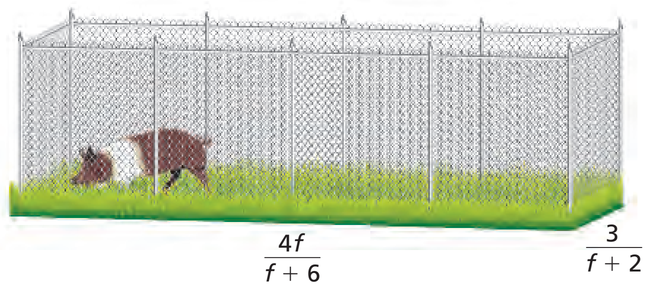
13. $\dfrac{3}{s-2} = \dfrac{4}{s}$

14. $2 = \dfrac{6}{2w+1}$

15. $-5 + \dfrac{2h}{h+2} = \dfrac{7h}{h+2}$

16. $\dfrac{2}{g} + \dfrac{5}{g(g+1)} = \dfrac{6}{g+1}$

17. PIGPEN You are installing a fence around a pigpen. Write an expression that represents the amount of fencing you need. *(Section 11.6)*



18. RAKING You can rake your front yard in 30 minutes. Your friend can rake the same yard in 50 minutes. Working together, how much time does it take to rake the yard? *(Section 11.7)*

Alternative Assessment Options

Math Chat	Student Reflective Focus Question
Structured Interview	Writing Prompt

Math Chat

- Have students work in pairs. Assign Quiz Exercises 13–16 to each pair. Each student works through all four problems. After the students have worked through the problems, they take turns talking through the processes that they used to get each answer. Students analyze and evaluate the mathematical thinking and strategies used.
- The teacher should walk around the classroom listening to the pairs and ask questions to ensure understanding.

Study Help Sample Answers

Remind students to complete Graphic Organizers for the rest of the chapter.

7.

Multiplying and Dividing Rational Expressions

Examples	Non-Examples
$\frac{2}{x}\cdot\frac{2}{x^3}=\frac{4}{x^4}$	$\frac{2}{x}\cdot\frac{2}{x^3}=4x^4$ ✗
$\frac{2}{x}\div\frac{x^3}{2}=\frac{2}{x}\cdot\frac{2}{x^3}=\frac{4}{x^4}$	$\frac{\cancel{2}}{x}\div\frac{x^3}{\cancel{2}}=\frac{1}{x}\cdot\frac{1}{x^3}=\frac{1}{x^4}$ ✗
$\frac{x^2-1}{x+2}\cdot\frac{x+2}{x-1}=\frac{(x+1)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{(x+2)}\cancel{(x-1)}}$ $=x+1$	$\frac{x^2-1}{x+2}\cdot\frac{x+2}{x-1}=\frac{x^3+2x^2-x-2}{x^2+x-2}$ (Correct, but not simplified)
$\frac{x^2-1}{x+2}\div\frac{x-1}{x+2}=\frac{x^2-1}{x+2}\cdot\frac{x+2}{x-1}$ $=\frac{(x+1)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{(x+2)}\cancel{(x-1)}}$ $=x+1$	$\frac{x^2-1}{x+2}\div\frac{x-1}{x+2}=\frac{x+2}{x^2-1}\cdot\frac{x-1}{x+2}$ ✗ $=\frac{\cancel{(x+2)}\cancel{(x-1)}}{(x+1)\cancel{(x-1)}\cancel{(x+2)}}$ $=\frac{1}{x+1}$

8–10. Available at *BigIdeasMath.com.*

Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Study Help • Practice A and Practice B • Puzzle Time Lesson Tutorials *BigIdeasMath.com*	Resources by Chapter • Enrichment and Extension • School-to-Work Game Closet at *BigIdeasMath.com* Start the Chapter Review

Answers

1. $\frac{2c(c+2)}{15}$
2. $\frac{b^2}{a}$
3. $\frac{k}{5}$
4. $\frac{m-6}{m+6}$
5. j^2+2j+3
6. $m-7$
7. $d-2+\frac{2}{d-3}$
8. $5n+5+\frac{12}{n-1}$
9. $-\frac{5}{v+1}$
10. $\frac{-r+1}{2r-3}$
11. $\frac{-t^2+5}{12t}$
12. $\frac{p}{(p-4)(p+5)}$
13. $s=8$
14. $w=1$
15. $h=-1$
16. $g=\frac{7}{4}$
17. $\frac{8f^2+22f+36}{(f+6)(f+2)}$
18. 18.75 minutes

Technology for the Teacher

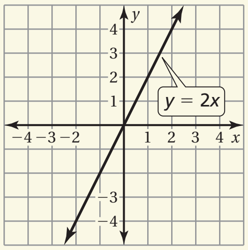
Online Assessment
Assessment Book
ExamView® Assessment Suite

For the Teacher
Additional Review Options

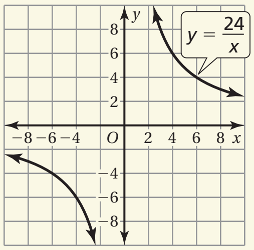
- *BigIdeasMath.com*
- Online Assessment
- Game Closet at *BigIdeasMath.com*
- Vocabulary Help
- Resources by Chapter

Answers

1. $y = 2x$



2. $y = \dfrac{24}{x}$



Review of Common Errors

- **Exercises 1–2** Students may substitute the wrong values for *x* and *y*. Tell them to be careful when substituting and that *k* should still be in the equation after substituting for *x* and *y*.

11 Chapter Review

Check It Out
Vocabulary Help
BigIdeasMath.com

Review Key Vocabulary

direct variation, *p. 544*
inverse variation, *p. 544*
rational function, *p. 552*
excluded value, *p. 552*
asymptote, *p. 553*
inverse relation, *p. 558*
inverse function, *p. 559*
rational expression, *p. 562*
simplest form of a rational expression, *p. 562*
least common denominator of rational expressions, *p. 583*
rational equation, *p. 590*

Review Examples and Exercises

11.1 Direct and Inverse Variation *(pp. 542–549)*

The variable y varies inversely with x. When $x = 3, y = 2$.

a. Write an inverse variation equation that relates x and y.

Find the value of k.

$$y = \frac{k}{x} \qquad \text{Write the inverse variation equation.}$$

$$2 = \frac{k}{3} \qquad \text{Substitute 3 for } x \text{ and 2 for } y.$$

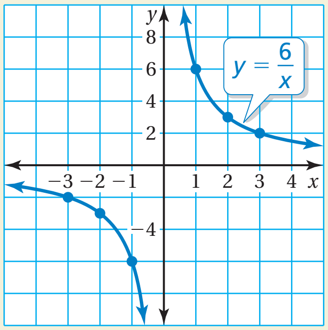
$$6 = k \qquad \text{Multiply each side by 3.}$$

So, an equation that relates x and y is $y = \frac{6}{x}$.

b. Graph the inverse variation equation. Describe the domain and range.

Make a table of values.

x	−3	−2	−1	0	1	2	3
y	−2	−3	−6	undef.	6	3	2



Plot the ordered pairs. Draw a smooth curve through the points in each quadrant. Both the domain and range are all real numbers except 0.

Exercises

1. The variable y varies directly with x. When $x = 6$, $y = 12$. Write and graph a direct variation equation that relates x and y.

2. The variable y varies inversely with x. When $x = 3$, $y = 8$. Write and graph an inverse variation equation that relates x and y.

11.2 Graphing Rational Functions (pp. 550–559)

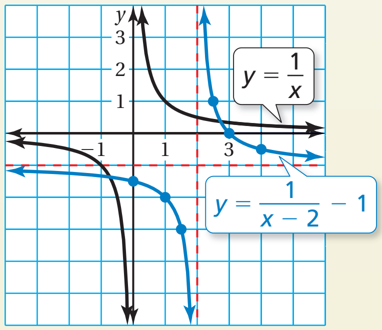
Graph $y = \frac{1}{x-2} - 1$. Compare the graph to the graph of $y = \frac{1}{x}$.

Step 1: Make a table of values. The vertical asymptote is $x = 2$, so choose x-values on either side of 2.

x	0	1	1.5	2	2.5	3	4
y	-1.5	-2	-3	undef.	1	0	-0.5

Step 2: Use dashed lines to graph the asymptotes $x = 2$ and $y = -1$. Then plot the ordered pairs.

Step 3: Draw a smooth curve through the points on each side of the vertical asymptote.



The graph of $y = \frac{1}{x-2} - 1$ is a translation 1 unit down and 2 units right of the graph of $y = \frac{1}{x}$.

Find the inverse of $f(x) = \frac{2}{x}$. Graph the inverse function.

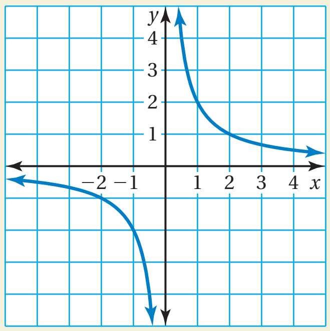
$y = \frac{2}{x}$ Replace $f(x)$ with y.

$x = \frac{2}{y}$ Switch x and y.

$xy = 2$ Multiply each side by y.

$y = \frac{2}{x}$ Divide each side by x.

$f^{-1}(x) = \frac{2}{x}$ Replace y with $f^{-1}(x)$.



Exercises

Graph the function. Compare the graph to the graph of $y = \frac{1}{x}$.

3. $y = \frac{1}{x+5}$

4. $y = \frac{1}{x} - 4$

5. $y = \frac{1}{x-7} + 1$

Find the inverse of the function. Graph the inverse function.

6. $f(x) = x + 2$

7. $f(x) = \frac{1}{2}x - 5$

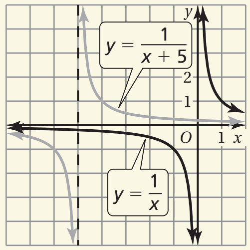
8. $f(x) = \frac{1}{x} + 7$

Review of Common Errors (continued)

- **Exercises 3–5** Students may not identify the horizontal and vertical asymptotes before trying to graph the function. Encourage them to use the asymptotes to help graph the function.
- **Exercises 6–8** Students may stop after switching x and y. Remind them to solve for y.

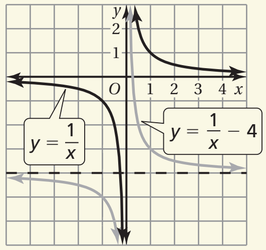
Answers

3.



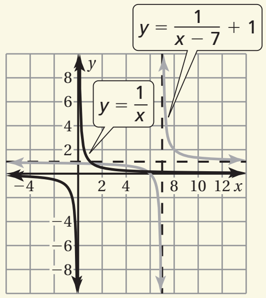
The graph of $y = \dfrac{1}{x+5}$ is a translation 5 units left of the graph of $y = \dfrac{1}{x}$.

4.



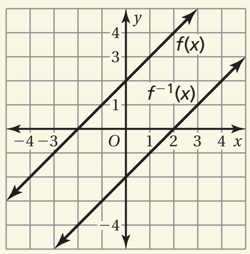
The graph of $y = \dfrac{1}{x} - 4$ is a translation 4 units down of the graph of $y = \dfrac{1}{x}$.

5.



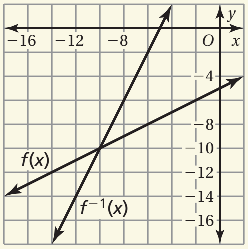
The graph of $y = \dfrac{1}{x-7} + 1$ is a translation 7 units right and 1 unit up of the graph of $y = \dfrac{1}{x}$.

6. $f^{-1}(x) = x - 2$

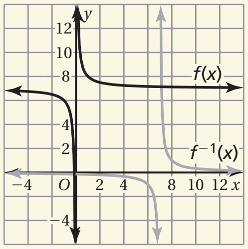


Answers

7. $f^{-1}(x) = 2x + 10$



8. $f^{-1}(x) = \dfrac{1}{x - 7}$



9. $\dfrac{9}{2z^2}$; $z = 0$

10. The expression is in simplest form; $n = 1$

11. $\dfrac{b + 3}{b - 5}$; $b = -6$, $b = 5$

Review of Common Errors (continued)

- **Exercises 9–11** Students may not state the correct excluded value(s). Remind them to use the original expression to find the excluded value(s).
- **Exercises 12–14** Students may not factor completely before simplifying the rational expression. Remind students to factor completely in order to divide out common factors so that the rational expression is in simplest form.
- **Exercises 13–14** Students may not rewrite the division as multiplication. Remind students that when they divide rational expressions, they need to multiply the dividend by the reciprocal of the divisor.
- **Exercise 16** Students may forget to insert missing terms before dividing the polynomials.
- **Exercises 19–21** Students may add the denominators as well as the numerators. Remind them that only the numerators are added when adding rational expressions with a common denominator.
- **Exercises 22–23** Students may multiply the numerators and multiply the denominators of the rational expressions instead of finding the cross products. Remind them of the Cross Products Property.
- **Exercise 24** Students may not multiply each side of the equation by the LCD. Remind them of the Multiplication Property of Equality.

11.3 Simplifying Rational Expressions (pp. 560–565)

Simplify $\frac{v^2 - 9}{v^2 - 3v}$, if possible. State the excluded value(s).

$$\frac{v^2 - 9}{v^2 - 3v} = \frac{(v-3)(v+3)}{v(v-3)}$$ Factor.

$$= \frac{\cancel{(v-3)}(v+3)}{v\cancel{(v-3)}}$$ Divide out the common factor.

$$= \frac{v+3}{v}$$ Simplify.

The excluded values are $v = 0$ and $v = 3$.

Exercises

Simplify the rational expression, if possible. State the excluded value(s).

9. $\frac{18z^2}{4z^4}$

10. $\frac{n^2 + 1}{n - 1}$

11. $\frac{b^2 + 9b + 18}{b^2 + b - 30}$

11.4 Multiplying and Dividing Rational Expressions (pp. 568–573)

Find the product or quotient.

a. $\frac{7x}{x+4} \cdot \frac{x+4}{x^2} = \frac{7x(x+4)}{x^2(x+4)}$ Multiply numerators and denominators.

$$= \frac{7\cancel{x}\cancel{(x+4)}}{x^{\cancel{2}}\cancel{(x+4)}}$$ Divide out the common factors.

$$= \frac{7}{x}$$ Simplify.

b. $\frac{t-6}{10} \div \frac{6-t}{12} = \frac{t-6}{10} \cdot \frac{12}{6-t}$ Multiply by the reciprocal.

$$= \frac{t-6}{10} \cdot \frac{12}{-(t-6)}$$ Rewrite $6 - t$ as $-(t - 6)$.

$$= \frac{12(t-6)}{-10(t-6)}$$ Multiply numerators and denominators.

$$= \frac{\overset{6}{\cancel{12}}\cancel{(t-6)}}{-\underset{5}{\cancel{10}}\cancel{(t-6)}}$$ Divide out the common factors.

$$= -\frac{6}{5}$$ Simplify.

Exercises

Find the product or quotient.

12. $\dfrac{9}{10r} \cdot \dfrac{5r^3}{6}$ **13.** $\dfrac{k+5}{6k^2} \div \dfrac{5+k}{12k}$ **14.** $\dfrac{h^2+8h}{h} \div (h^2+7h-8)$

11.5 Dividing Polynomials *(pp. 574–579)*

Find $(-2x^2 + 8x + 1) \div 2x$.

$$
\begin{aligned}
(-2x^2+8x+1) \div 2x &= \frac{-2x^2+8x+1}{2x} && \text{Write as a fraction.}\\
&= \frac{-2x^2}{2x} + \frac{8x}{2x} + \frac{1}{2x} && \text{Divide each term by } 2x.\\
&= \frac{-\cancel{2}x^{\cancel{2}}}{\cancel{2}\cancel{x}} + \frac{\overset{4}{\cancel{8}}\cancel{x}}{\cancel{2}\cancel{x}} + \frac{1}{2x} && \text{Divide out the common factors.}\\
&= -x + 4 + \frac{1}{2x} && \text{Simplify.}
\end{aligned}
$$

Find $(z^2 - 2z - 5) \div (z + 3)$.

Step 1: Divide the first term of the dividend by the first term of the divisor.

Align like terms in the quotient and dividend.

$$
\begin{array}{rl}
z\phantom{{}-2z-5} & \text{Divide: } z^2 \div z = z.\\
z+3\,\overline{)\,z^2-2z-5} & \\
\underline{z^2+3z\phantom{{}-5}} & \text{Multiply: } z(z+3).\\
-5z-5 & \text{Subtract. Bring down the } -5.
\end{array}
$$

Step 2: Divide the first term of $-5z - 5$ by the first term of the divisor.

$$
\begin{array}{rl}
z-5 & \text{Divide: } -5z \div z = -5.\\
z+3\,\overline{)\,z^2-2z-5} & \\
\underline{z^2+3z\phantom{{}-5}} & \\
-5z-5 & \\
\underline{-5z-15} & \text{Multiply: } -5(z+3).\\
10 & \text{Subtract.}
\end{array}
$$

So, $(z^2 - 2z - 5) \div (z + 3) = z - 5 + \dfrac{10}{z+3}$.

Exercises

Find the quotient.

15. $(8n^3 + 3n) \div 2n^2$

16. $(b^2 - 36) \div (b - 6)$

17. $(x^2 + 6x + 3) \div (x + 2)$

18. $(4c - 1) \div (c + 5)$

Review Game

Review for You

Materials per Group

- copy of the Chapter Review from the Pupil's Edition
- chalk or dry erase marker
- eraser

Directions

Divide the class into four teams. Team members gather in groups at the board.

One member of each team works the first problem from the Chapter Review. Coaching from the other team members is allowed.

Check each student's work and award 2 points to the team if the work is correct. If the work needs to be corrected, award 1 point to the team after the student makes the corrections.

Team members take turns working the problems. Repeat the process until you finish the Chapter Review or until you run out of time.

Who wins?

The team with the most points wins.

Variations

- Have students write review problems on index cards. Draw a card for students to use.
- To provide a cumulative review, write problems on index cards from anywhere in the book. Have team members select a card at random.

For the Student Additional Practice

- Lesson Tutorials
- Multi-Language Glossary
- Self-Grading Progress Check
- *BigIdeasMath.com*
  Dynamic Student Edition
  Student Resources

Answers

12. $\frac{3r^2}{4}$

13. $\frac{2}{k}$

14. $\frac{1}{h-1}$

15. $4n + \frac{3}{2n}$

16. $b + 6$

17. $x + 4 - \frac{5}{x+2}$

18. $4 - \frac{21}{c+5}$

19. $\frac{2h-9}{h-10}$

20. $\frac{13x+5}{x(x+1)}$

21. $\frac{12}{(x+8)(x-7)}$

22. no solution

23. $y = -3, y = 6$

24. $t = -\frac{4}{3}$

25. 10 first serves

My Thoughts on the Chapter

Teacher Tip

Not allowed to write in your teaching edition? Use sticky notes to record your thoughts.

What worked. . .

What did not work. . .

What I would do differently. . .

11.6 Adding and Subtracting Rational Expressions (pp. 580–587)

Find the difference $\dfrac{y+3}{7y^2} - \dfrac{4}{5y}$.

$$\frac{y+3}{7y^2} - \frac{4}{5y} = \frac{(y+3)(5)}{7y^2(5)} - \frac{4(7y)}{5y(7y)} \qquad \text{Rewrite using the LCD, } 35y^2.$$

$$= \frac{5y+15}{35y^2} - \frac{28y}{35y^2} \qquad \text{Simplify.}$$

$$= \frac{-23y+15}{35y^2} \qquad \text{Subtract the numerators.}$$

Exercises

Find the sum or difference.

19. $\dfrac{5h-2}{h-10} - \dfrac{3h+7}{h-10}$

20. $\dfrac{5}{x} + \dfrac{8}{x+1}$

21. $\dfrac{4-x}{x^2+x-56} + \dfrac{1}{x-7}$

11.7 Solving Rational Equations (pp. 588–593)

You own a farm in a computer game. Twenty of the 120 animals on your farm are cows. You buy cows and increase the ratio of cows to total animals to 1 : 5. How many cows do you buy?

Write an equation for the ratio of cows to total animals after buying x cows.

Cows

Total animals

$$\frac{20+x}{120+x} = \frac{1}{5}$$

New ratio of cows to total animals

$$5(20+x) = 120+x \qquad \text{Cross Products Property}$$

$$100+5x = 120+x \qquad \text{Distributive Property}$$

$$4x = 20 \qquad \text{Simplify.}$$

$$x = 5 \qquad \text{Divide each side by 4.}$$

You buy 5 cows.

Exercises

Solve the equation. Check your solution(s).

22. $\dfrac{1}{x+2} = \dfrac{12}{3x+6}$

23. $\dfrac{9}{y+6} = \dfrac{y}{y+2}$

24. $\dfrac{5}{t} - \dfrac{3}{4} = \dfrac{6}{t}$

25. **TENNIS** A tennis player lands 25 out of 40 first serves in bounds for a success rate of 62.5%. How many more consecutive first serves must she land in bounds to increase her success rate to 70%?

11 Chapter Test

Check It Out
Test Practice
BigIdeasMath.com

1. The variable y varies directly with x. When $x = 3$, $y = 18$. Write and graph a direct variation equation that relates x and y.

2. The variable y varies inversely with x. When $x = 6$, $y = 4$. Write and graph an inverse variation equation that relates x and y.

Graph the function. Compare the graph to the graph of $y = \frac{1}{x}$.

3. $y = \dfrac{1}{x-6}$ **4.** $y = \dfrac{1}{x} + 3$ **5.** $y = \dfrac{1}{x+4} - 5$

Find the inverse of the function. Graph the inverse function.

6. $f(x) = x - 7$ **7.** $f(x) = \dfrac{1}{5}x - 7$ **8.** $f(x) = \dfrac{3}{x+4}$

Simplify.

9. $\dfrac{y^2 + 5y - 24}{y^2 + 10y + 16}$ **10.** $\dfrac{8r^4}{5} \cdot \dfrac{15r}{6r^3}$ **11.** $\dfrac{x^2 - 25}{x+5} \div (x^2 - 3x - 10)$

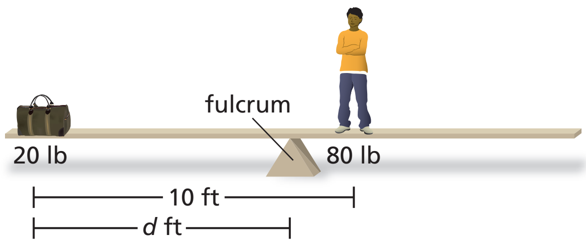
12. $\dfrac{6k+1}{2k-4} + \dfrac{2k+3}{2k-4}$ **13.** $\dfrac{4}{p+6} - \dfrac{3}{p}$ **14.** $\dfrac{18z+27}{z^2+3z-54} + \dfrac{z}{z+9}$

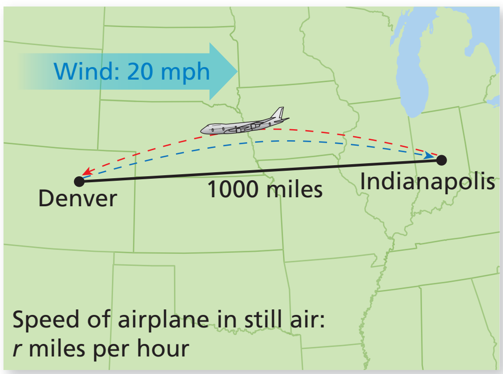
15. Find $(12d^3 + 8d - 6) \div 3d^2$. **16.** Find $(b^2 - 4b + 10) \div (b + 3)$.

Solve the equation. Check your solution(s).

17. $\dfrac{1}{x-5} = \dfrac{3}{2x+7}$ **18.** $\dfrac{a}{a+3} = \dfrac{4}{a+5}$ **19.** $\dfrac{6}{n} - \dfrac{2}{n-3} = \dfrac{5}{n-3}$

20. **BALANCE** To balance the board in the diagram, the distance (in feet) of each object from the center of the board must vary inversely with its weight (in pounds). What is the distance of the suitcase from the fulcrum?





21. **AIRPLANE** An airplane makes a round trip between two cities. The airplane flies with the wind when heading east and against the wind when heading west. Write an expression for the total time of the trip.

22. **DELIVERY TRUCK** Working alone, it takes you 30 minutes, your friend 30 minutes, and your supervisor 15 minutes to unload a delivery truck. Working together, how much time does it take all three of you to unload the truck?

Test Item References

Chapter Test Questions	Section to Review	Common Core State Standards
1, 2, 21	11.1	A.REI.10
3–8	11.2	A.REI.10, F.BF.4a
9, 10	11.3	A.SSE.2
11, 12	11.4	A.SSE.2
13, 14	11.5	A.SSE.2
15–17, 22	11.6	A.SSE.2
18–20, 23	11.7	A.CED.1

Test-Taking Strategies

Remind students to quickly look over the entire test before they start so that they can budget their time. Have students use the **Stop** and **Think** strategy before they answer each question.

Common Errors

- **Exercises 1 and 2** Students may substitute the wrong values for *x* and *y*. Tell them to be careful when substituting and that *k* should still be in the equation after substituting for *x* and *y*.
- **Exercises 3–5** Students may not identify the horizontal and vertical asymptotes before trying to graph the function. Encourage them to use the asymptotes to help graph the function.
- **Exercises 11–12** Students may not factor completely before simplifying the rational expression. Remind students to factor completely in order to divide out common factors so that the rational expression is in simplest form.
- **Exercises 18–19** Students may multiply the numerators and multiply the denominators of the rational expressions instead of finding the cross products. Remind them of the Cross Products Property.

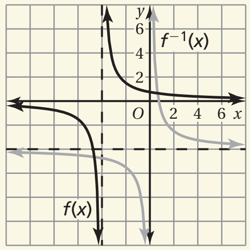
Reteaching and Enrichment Strategies

If students need help. . .	If students got it. . .
Resources by Chapter • Practice A and Practice B • Puzzle Time Record and Practice Journal Practice Differentiating the Lesson Lesson Tutorials *BigIdeasMath.com* Skills Review Handbook	Resources by Chapter • Enrichment and Extension • School-to-Work • Financial Literacy • Technology Connection • Life Connections Game Closet at *BigIdeasMath.com* Start Standards Assessment

Answers

1–7. See Additional Answers.

8. $f^{-1}(x) = \dfrac{3}{x} - 4$



9. $\dfrac{y-3}{y+2}$

10. $4r^2$

11. $\dfrac{1}{x+2}$

12. $\dfrac{4k+2}{k-2}$

13. $\dfrac{p-18}{p(p+6)}$

14. $\dfrac{z+3}{z-6}$

15. $4d + \dfrac{8}{3d} - \dfrac{2}{d^2}$

16. $b - 7 + \dfrac{31}{b+3}$

17. $x = 22$

18. $a = -4, a = 3$

19. $n = -18$

20. 8 ft

21. $\dfrac{2000r}{(r+20)(r-20)}$

22. 7.5 minutes

Technology for the Teacher

Online Assessment
Assessment Book
ExamView® Assessment Suite

Test Taking Strategies

Available at *BigIdeasMath.com*

After Answering Easy Questions, Relax
Answer Easy Questions First
Estimate the Answer
Read All Choices before Answering
Read Question before Answering
Solve Directly or Eliminate Choices
Solve Problem before Looking at Choices
Use Intelligent Guessing
Work Backwards

About this Strategy

When taking a multiple choice test, be sure to read each question carefully and thoroughly. One way to answer the question is to work backwards. Try putting the responses into the question, one at a time and see if you get a correct solution.

Answers

1. C
2. H
3. 2
4. B

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Item Analysis

1. **A.** The student incorrectly subtracts 3 instead of adding 3 to represent a translation 3 units up.

   B. The student switches the roles of h and k as well as their signs in writing an equation of the form $y = \dfrac{1}{x - h} + k$.

   C. Correct answer

   D. The student incorrectly subtracts 3 instead of adding 3 to represent a translation 3 units up, and subtracts 2 instead of adding 2 to represent a translation 2 units left.

2. **F.** The student incorrectly uses the square root of the sum of the legs.

   G. The student incorrectly uses the square root of the square of the longer leg.

   H. Correct answer

   I. The student estimates incorrectly.

3. **Gridded response:** Correct answer: 2

   Common error: The student incorrectly calculates the value of $\dfrac{1}{1/2}$ as $\dfrac{1}{2}$.

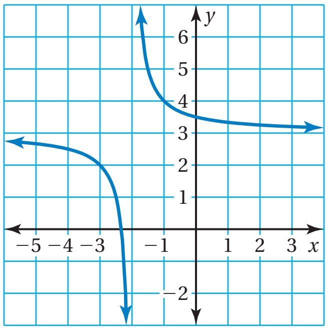
4. **A.** The student multiplies the expression by $\dfrac{-1}{-1}$ and incorrectly places a negative sign with the result.

   B. Correct answer

   C. The student does not recognize that because the numerator and denominator are opposites, they have a common factor.

   D. The student fails to realize that because the numerator and denominator are opposites, the result of dividing out the common factor is -1.

1. Which function is shown by the graph? *(A.REI.10)*



A. $y = \dfrac{1}{x+2} - 3$

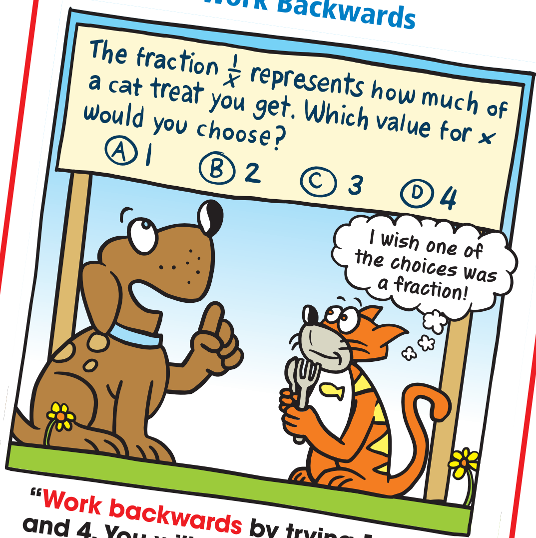
B. $y = \dfrac{1}{x+3} + 2$

C. $y = \dfrac{1}{x+2} + 3$

D. $y = \dfrac{1}{x-2} - 3$

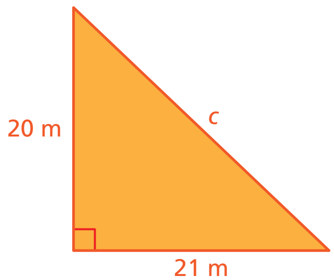
Test-Taking Strategy

Work Backwards



"**Work backwards** by trying 1, 2, 3, and 4. You will see that $x = 1$ gives you a whole treat."

2. What is the value of c in the triangle shown? *(8.G.7)*



F. $\sqrt{41}$ m

G. 21 m

H. 29 m

I. 30 m

3. What is the y-coordinate of the focus of the graph of $y = \frac{1}{8}x^2$? *(F.IF.4)*

4. What is the simplest form of the rational expression $\dfrac{4x-3}{3-4x}$? *(A.SSE.2)*

A. $-\dfrac{3-4x}{4x-3}$

B. -1

C. $\dfrac{4x-3}{3-4x}$

D. 1

5. What is the solution of the system of equations? *(A.REI.7)*

$$y = x^2 + 2x - 7$$
$$y = 2x - 7$$

F. $(-3, -4)$

G. $(0, -7)$

H. $(-7, 0)$

I. no real solutions

6. What are the solutions of the equation? *(A.SSE.3a)*

$$x^4 - 2x^3 - 3x^2 = 0$$

A. $x = 0$

B. $x = 1, x = -3$

C. $x = 0, x = -1, x = 3$

D. $x = 0, x = 3$

7. What is the difference of the rational expressions? *(A.SSE.2)*

$$\frac{5}{3x} - \frac{3}{4x}$$

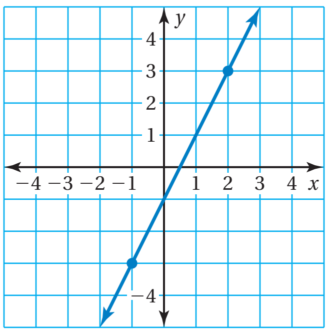
F. $-\frac{2}{x}$

G. $\frac{2}{3x}$

H. $\frac{2}{x}$

I. $\frac{11}{12x}$

8. What is the slope of the line shown in the graph? *(F.IF.6)*



9. What is an equation for the *n*th term of the geometric sequence? *(F.BF.2)*

n	1	2	3	4
a_n	4	8	16	32

A. $a_n = 4(2)^{n-1}$

B. $a_n = 4^{n-1}$

C. $a_n = 2(4)^{n-1}$

D. $a_n = 2^{n-1}$

Item Analysis (continued)

5. **F.** The student uses a random solution point of the first equation.
   G. Correct answer
   H. The student reverses the coordinates of the solution.
   I. The student graphs the equations and they do not intersect due to graphing error.

6. **A.** The student notices that the equation is true when 0 is substituted for x and fails to consider other solutions.
   B. The student starts by factoring out x^2, then forgets about the x^2 after incorrectly factoring the remaining quadratic expression.
   C. Correct answer
   D. The student graphs the related function for positive values of x and does not realize that $x = -1$ is also a solution.

7. **F.** The student subtracts both the numerators and the denominators.
   G. The student makes an error in rewriting the second term using the LCD.
   H. The student subtracts both the numerators and the denominators and omits the negative sign.
   I. Correct answer

8. **Gridded response:** Correct answer: 2

   Common error: The student calculates the rise between the two points, then uses the reverse order of the two points to calculate the run, yielding a slope of -2.

9. **A.** Correct answer
   B. The student uses the form $a_n = a_1^{n-1}$ instead of $a_n = a_1 r^{n-1}$.
   C. The student uses the form $a_n = ra_1^{n-1}$ instead of $a_n = a_1 r^{n-1}$.
   D. The student uses the form $a_n = r_1^{n-1}$ instead of $a_n = a_1 r^{n-1}$.

Answers

5. G
6. C
7. I
8. 2
9. A

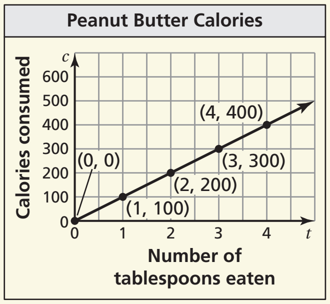
Answers

10. *Part A:* Independent variable: t; dependent variable: c

Part B:

Input, t	Output, c
0	0
1	100
2	200
3	300
4	400

Part C:



Part D: continuous; Because you can eat any part of a tablespoon of peanut butter, t can be any value greater than or equal to 0.

11. H

12. B

13. G

Answer for Extra Example

1. **A.** Correct answer

   B. The student makes a sign error.

   C. The student takes the reciprocal of the rational expression.

   D. The student takes the reciprocal of the rational expression and uses addition, the inverse operation of subtraction.

Item Analysis (continued)

10. **4 points** The student demonstrates a thorough understanding of discrete and continuous functions, making a correct table, graph, description of the variables, and explanation of why the domain is continuous.

3 points The student demonstrates an essential but less than thorough understanding of discrete and continuous functions, with some part of making a table, graph, description of the variables, or the explanation incorrect or incomplete.

2 points The student demonstrates a partial understanding of discrete and continuous functions, making a few mistakes. The student may call the function discrete, thinking that only whole number values of t are possible.

1 point The student demonstrates limited understanding of discrete and continuous functions. The student's work is incomplete or exhibits many flaws.

0 points The student provides no response, a completely incorrect or incomprehensible response, or a response that demonstrates insufficient understanding.

11. **F.** The student forgets to change the direction of the inequality sign when multiplying both sides by -3.

G. The student forgets to change the direction of the inequality sign and incorrectly multiplies two negative numbers.

H. Correct answer

I. The student incorrectly multiplies two negative numbers.

12. **A.** The student misunderstands the process of dividing rational expressions.

B. Correct answer

C. The student fails to recognize that a common factor of x can be divided out of the numerator and denominator.

D. The student misunderstands the process of dividing rational expressions.

13. **F.** The student reverses the coordinates.

G. Correct answer

H. The student reverses the coordinates and omits a negative sign.

I. The student omits a negative sign.

Extra Example

1. What is the inverse of the function $f(x) = \frac{1}{x-4}$? *(F.BF.4a)*

A. $f^{-1}(x) = \frac{1}{x} + 4$

B. $f^{-1}(x) = \frac{1}{x} - 4$

C. $f^{-1}(x) = x - 4$

D. $f^{-1}(x) = x + 4$

10. One tablespoon of peanut butter contains 100 calories. The number *c* of calories consumed is a function of the number *t* of tablespoons of peanut butter eaten. *(F.IF.1)*

Think Solve Explain

Part A Identify the independent and dependent variables.

Part B Make an input-output table.

Part C Graph the function.

Part D Is the domain discrete or continuous? Explain.

11. What is the solution of the inequality shown below? *(A.REI.3)*

$$\frac{y}{-3} - 4 > -12$$

F. $y > 24$

G. $y > -24$

H. $y < 24$

I. $y < -24$

12. John was finding the quotient of the rational expressions in the box below. *(A.SSE.2)*

$$\frac{3x}{x-4} \div \frac{2x}{4-x} = \frac{3x}{x-4} \bullet \frac{4-x}{2x}$$

$$= \frac{3x\cancel{(4-x)}}{2x\cancel{(x-4)}}$$

$$= \frac{3x}{2x}$$

$$= \frac{3}{2}$$

What should John do to correct the error that he made?

A. Do not multiply by the reciprocal.

B. Factor out -1 from $(4 - x)$ before dividing out common factors.

C. Divide $3x$ by $2x$ to get an answer of x.

D. Use long division to find the quotient.

13. What is the solution of the system of linear equations shown below? *(A.REI.6)*

$$y = 2x - 1$$

$$y = 3x + 5$$

F. $(-13, -6)$

G. $(-6, -13)$

H. $(-13, 6)$

I. $(-6, 13)$